CHARACTER SHEAVES ON DISCONNECTED GROUPS, IX

G. Lusztig

Introduction

Throughout this paper, G denotes a fixed, not necessarily connected, reductive algebraic group over an algebraically closed field $\mathbf{k}$. This paper is a part of a series [L9] which attempts to develop a theory of character sheaves on G.

One of the main constructions in [L3] (going back to [L14]) was a procedure which to any character sheaf on G^0 associates a certain two-sided cell in an (extended) Coxeter group. A variant of this construction (restricted to "unipotent" character sheaves) was later given by Grojnowski [Gr]. Here we give a construction which generalizes that in [L3] (and takes into account the approach in [Gr]) which to any (parabolic) character sheaf on $Z_{J,D}$ associates a certain type of two-sided cell.

The paper is organized as follows. In Section 40 we study certain equivariant sheaves on $G^0/U^* \times G^0/U^*$ (where U^* is the unipotent radical of a Borel in G^0) under the convolution operation. Some results in this section are implicit in [L14, Ch.1]. In Section 41 we study the character sheaves on $Z_{\emptyset,D}$ (where D is a connected component of G) by connecting them with sheaves on $G^0/U^* \times G^0/U^*$. We use this study to attach a two-sided cell to any character sheaf on $Z_{J,D}$. (See 41.4.) In Section 42 we study the interaction between the duality operation $\mathbf{d}$ (see 38.10, 38.11) and the functor $\mathfrak{f}_{\emptyset,\mathbf{I}}$ (see 36.4). The main result in this section is Proposition 42.9 which contains [L3, III, Cor.15.8(b)] as a special case (with $G = G^0, v = 1$).

Notation We fix a 1-dimensional $\bar{\mathbf{Q}}_l$-vector space V with a given isomorphism $V^{\otimes 2} \xrightarrow{\sim} \bar{\mathbf{Q}}_l(1)$ (Tate twist of $\bar{\mathbf{Q}}_l$). For $n \in \mathbf{N}$ we set $\bar{\mathbf{Q}}_l(n/2) = V^{\otimes n}$. For $n \in \mathbf{Z}, n < 0$ let $\bar{\mathbf{Q}}_l(n/2)$ be the dual space of $\bar{\mathbf{Q}}_l(-n/2)$. If X is an algebraic variety and $A \in \mathcal{D}(X), n \in \mathbf{Z}$ we write $A[[n/2]]$ instead of $A[n](n/2)$. (When n is even this agrees with the notation in [L9, II, p.73].)

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42. Duality and the functor $\mathfrak{f}_{\emptyset,\mathbf{I}}$.

40. Sheaves on $G^0/U^* \times G^0/U^*$

40.1. Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$. Let $\hat{H}$ (resp. H) be the $\mathcal{A}$-module consisting of all formal (resp. finite) linear combinations $\sum_{w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}} a_{w,\lambda}\tilde{T}_w 1_\lambda$ with $a_{w,\lambda} \in \mathcal{A}$. Note that H is naturally an $\mathcal{A}$-submodule of $\hat{H}$ with $\mathcal{A}$-basis $\{\tilde{T}_w 1_\lambda; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}\}$. For any $n \in \mathbf{N}^*_{\mathbf{k}}$, the $\mathcal{A}$-submodule of H spanned by $\{\tilde{T}_w 1_\lambda; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n\}$ may be naturally identified with H_n (see 31.2(a)). There is a unique $\mathcal{A}$-algebra structure on $\hat{H}$ in which the product of two elements

$h = \sum_{w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}} a_{w,\lambda}\tilde{T}_w 1_\lambda$, $h' = \sum_{w'\in\mathbf{W},\lambda'\in\underline{\mathfrak{s}}} a'_{w',\lambda'}\tilde{T}_{w'} 1_{\lambda'}$

as above is $hh' = \sum_{y\in\mathbf{W},\nu\in\underline{\mathfrak{s}}} b_{y',\nu}\tilde{T}_y 1_\nu$ where for any $\nu \in \underline{\mathfrak{s}}$,

$\sum_{w,w'\in\mathbf{W}} a_{w,w'^{-1}\nu} a'_{w',\nu}\tilde{T}_w\tilde{T}_{w'} 1_\nu = \sum_{y\in\mathbf{W}} b_{y,\nu}\tilde{T}_y 1_\nu$

is computed in the algebra structure of H_n for any n such that $\nu \in \underline{\mathfrak{s}}_n$. Thus $\hat{H}$ becomes an associative algebra with 1; H is a subalgebra (without 1) and, for $n \in \mathbf{N}^*_{\mathbf{k}}$, H_n is a subalgebra (with a different 1) with the algebra structure as in 31.2.

Now in the definition of $\hat{H}$ given above, although $\tilde{T}_w 1_\lambda$ is defined, the elements $\tilde{T}_w, 1_\lambda$ are not defined separately (as was the case in H_n). To remedy this we set $\tilde{T}_w = \sum_{\lambda\in\underline{\mathfrak{s}}} \tilde{T}_w 1_\lambda \in \hat{H}$ (for $w \in \mathbf{W}$) and $1_\lambda = \tilde{T}_1 1_\lambda \in H$ (for $\lambda \in \underline{\mathfrak{s}}$). Then $\tilde{T}_w 1_\lambda$ is the product of $\tilde{T}_w, 1_\lambda$ in the algebra $\hat{H}$. Note that $\tilde{T}_1$ is the unit element of $\hat{H}$ and the following equalities hold in $\hat{H}$:

$1_\lambda 1_\lambda = 1_\lambda$ for $\lambda \in \underline{\mathfrak{s}}, 1_\lambda 1_{\lambda'} = 0$ for $\lambda \neq \lambda'$ in $\underline{\mathfrak{s}}$;
$\tilde{T}_w\tilde{T}_{w'} = \tilde{T}_{ww'}$ for $w, w' \in \mathbf{W}$ such that $l(ww') = l(w) + l(w')$;
$\tilde{T}_w 1_\lambda = 1_{w\lambda}\tilde{T}_w$ for $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}$;
$\tilde{T}_s^2 = \tilde{T}_1 + (v - v^{-1})\sum_{\lambda\in\underline{\mathfrak{s}}; s\in\mathbf{W}_\lambda} \tilde{T}_s 1_\lambda$ for $s \in \mathbf{I}$.

By a standard argument we see that

(a) H is exactly the $\mathcal{A}$-algebra defined by the generators $\tilde{T}_w 1_l$ ($w \in \mathbf{W}$, $\lambda \in \underline{\mathfrak{s}}$) and the relations:

$(\tilde{T}_w 1_\lambda)(\tilde{T}_{w'} 1_{\lambda'}) = 0$ if $w, w' \in \mathbf{W}, \lambda, \lambda' \in \underline{\mathfrak{s}}, w'\lambda' \neq \lambda$,
$(\tilde{T}_w 1_{w'\lambda'}(\tilde{T}_{w'} 1_{\lambda'}) = \tilde{T}_{ww'} 1_{\lambda'}$ if $w, w' \in \mathbf{W}, \lambda, \lambda' \in \underline{\mathfrak{s}}$, $l(ww') = l(w) + l(w')$,
$(\tilde{T}_s 1_{s\lambda'})(\tilde{T}_s 1_{\lambda'}) = \tilde{T}_1 1_{\lambda'} + (v - v^{-1})c\tilde{T}_s 1_{\lambda'}$ if $s \in \mathbf{I}$, $\lambda' \in \underline{\mathfrak{s}}$ where $c = 1$ for $s \in \mathbf{W}_{\lambda'}$ and $c = 0$ for $s \notin \mathbf{W}_{\lambda'}$.

40.2. Let R, R^+ be as in 28.3. The following result is well known:

(a) *If $w \in \mathbf{W}, \alpha \in R^+$ and s_α is as in 28.3 then we have $l(ws_\alpha) > l(w)$ if and only if $w(\alpha) \in R^+$.*

Let $\lambda \in \underline{\mathfrak{s}}$. Let $R_\lambda, R^+_\lambda, \mathbf{W}_\lambda, H_\lambda$ be as in 34.2. We write $\vee_\lambda$ instead of $\vee^D_\lambda$ (as in 34.4 with $D = G^0$). We show:

(b) *If $w \in \mathbf{W}$ then $w\mathbf{W}_\lambda$ contains a unique element w_1 of minimal length; it is characterized by the property $w_1(R^+_\lambda) \subset R^+$.*

Let w_1 be an element of minimal length in $w\mathbf{W}_\lambda$. Let $\alpha \in R^+_\lambda$. Then $l(w_1 s_\alpha) \geq$

$l(w_1)$. Since $l(w_1 s_\alpha) = l(w_1) + 1 \mod 2$ we see that $l(w_1 s_\alpha) > l(w_1)$. By (a) we have $w_1(\alpha) \in R^+$. Thus, $w_1(R^+_\lambda) \subset R^+$. Now let $u \in \mathbf{W}_\lambda - \{1\}$. We pick $\alpha \in R^+_\lambda$ such that $u(\alpha)^{-1} \in R^+_\lambda$; then $w_1 u(\alpha)^{-1} \in R^+$. If $w_1 u$ has minimal length in $w\mathbf{W}_\lambda$ then by an earlier part of the argument applied to $w_1 u$ instead of w_1 we have $w_1 u(\alpha) \in R^+$, a contradiction. We see that w_1 is the unique element of minimal length in $w\mathbf{W}_\lambda$. It remains to show that if $u \in \mathbf{W}_\lambda$ satisfies $w_1 u(R^+_\lambda) \subset R^+$ then $u = 1$. If $u \neq 1$ then by an earlier part of the argument we have $w_1 u(\alpha)^{-1} \in R^+$ for some $\alpha \in R^+_\lambda$, a contradiction. This proves (b).

We show:

(c) *If $s \in \mathbf{I}$ and $w \in \mathbf{W}$ has minimal length in $w\mathbf{W}_\lambda$ then either (i) sw has minimal length in $sw\mathbf{W}_\lambda$ or (ii) $w^{-1}sw \in \mathbf{W}_\lambda$.*

There is a unique $\beta \in R^+$ such that $s(\beta)^{-1} \in R^+$. Assume that (i) does not hold. By (b) there exists $\alpha \in R^+_\lambda$ such that $sw(\alpha)^{-1} \in R^+$; moreover, $w(\alpha) \in R^+$. Hence $w(\alpha) = \beta$. We have $w^{-1}(\beta) = \alpha \in R_\lambda$ hence $w^{-1}sw \in \mathbf{W}_\lambda$ and (ii) holds. This proves (c).

For $z \in \mathbf{W}_\lambda$ let $\tilde{T}^\lambda_z, c^\lambda_z \in H_\lambda$ be as in 34.2 . Then $c^\lambda_z = \sum_{z' \in \mathbf{W}_\lambda} p^\lambda_{z',z} \tilde{T}^\lambda_{z'}$ where $p^\lambda_{z',z} \in \mathbf{Z}[v^{-1}]$ are uniquely defined.

For any $w \in \mathbf{W}$, $\lambda \in \underline{\mathfrak{s}}$ there is a unique element element of H which is equal to $c_{w,\lambda} \in H_n$ (see 34.4) for any n such that $\lambda \in \underline{\mathfrak{s}}_n$; we denote this element again by $c_{w,\lambda}$. We have

$c_{w,\lambda} = \sum_{w' \in \mathbf{W}} \pi_{w',w,\lambda} \tilde{T}_{w'} 1_\lambda$

where $\pi_{w',w,\lambda} \in \mathbf{Z}[v^{-1}]$ are uniquely defined. Note that

$\{c_{w,\lambda}; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}\}$ *is an $\mathcal{A}$-basis of H.*

We show:

(d) *Let $w, w' \in \mathbf{W}$. We write $w = w_1 z, w' = w'_1 z'$ where w_1 has minimal length in $w\mathbf{W}_\lambda$, w'_1 has minimal length in $w'\mathbf{W}_\lambda$ and $z, z' \in \mathbf{W}_\lambda$. If $w_1 \neq w'_1$ then $\pi_{w',w,\lambda} = 0$. If $w_1 = w'_1$ then $\pi_{w',w,\lambda} = p^\lambda_{z',z}$.*

From the definitions we see that if $w\lambda \neq w'\lambda$ then $\pi_{w',w,\lambda} = 0$. Thus we may assume that $w\lambda = w'\lambda$. We choose a sequence $s_1, s_2, \dots, s_r$ in $\mathbf{I}$ such that $w\lambda = w'\lambda = s_r s_{r-1} \dots s_1 \lambda \neq s_{r-1} \dots s_1 \lambda \neq \dots \neq s_1 \lambda \neq \lambda$.

We show that for $k \in [0, r]$, $s_k s_{k-1} \dots s_1$ has minimal length in $s_k s_{k-1} \dots s_1 \mathbf{W}_\lambda$. We argue by induction. For $k = 0$ the result is obvious. Assume now that $k \in [1, r]$. Since $s_{k-1} \dots s_1$ has minimal length in $s_{k-1} \dots s_1 \mathbf{W}_\lambda$ and $s_k s_{k-1} \dots s_1 \lambda \neq s_{k-1} \dots s_1 \lambda$ we see from (c) that $s_k s_{k-1} \dots s_1$ has minimal length in $s_k s_{k-1} \dots s_1 \mathbf{W}_\lambda$ as required.

In particular, $s_r s_{r-1} \dots s_1$ has minimal length in $s_r s_{r-1} \dots s_1 \mathbf{W}_\lambda$. Since $w\lambda = s_r s_{r-1} \dots s_1 \lambda$ we have $w = s_r s_{r-1} \dots s_1 h_1 h_2$ where $h_1 \in \vee_\lambda$, $h_2 \in \mathbf{W}_\lambda$. Then both w_1 and $s_r s_{r-1} \dots s_1 h_1$ have minimal length in $s_r s_{r-1} \dots s_1 h_1 \mathbf{W}_\lambda = w\mathbf{W}_\lambda = w_1 \mathbf{W}_\lambda$; using (b) we deduce that $s_r s_{r-1} \dots s_1 h_1 = w_1$. Hence $s_1 \dots s_r w = s_1 \dots s_r w_1 z = h_1 z$. Similarly, $s_1 \dots s_r w' = h'_1 z'$ where $h'_1 \in \vee_\lambda$.

From the results in 34.7-34.10 we see that $\pi_{w',w,\lambda} = p^\lambda_{s_1 \dots s_r w', s_1 \dots s_r w} = p^\lambda_{h'_1 z', h_1 z}$. Using $h_1, h'_1 \in \vee_\lambda$ and the definitions (34.2) we see that $p^\lambda_{h'_1 z', h_1 z} = 0$ if $h_1 \neq h'_1$

and $p^\lambda_{h'_1z',h_1z}=p^\lambda_{z',z}$ if $h_1=h'_1$.

It remains to show that we have $w_1=w'_1$ if and only if $h_1=h'_1$. We have $s_rs_{r-1}\dots s_1=h_1^{-1}w_1$ and similarly $s_rs_{r-1}\dots s_1=(h'_1)^{-1}w'_1$. Hence $h_1^{-1}w_1=(h'_1)^{-1}w'_1$. We see that $w_1=w'_1$ if and only if $h_1=h'_1$. This proves (d).

For $w'\le w$ in $\mathbf{W}$, $\lambda\in\underline{\mathfrak{s}}$ and $i\in\mathbf{Z}$ we define $N_{i,w',w,\lambda}\in\mathbf{Z}$ by

(e) $\pi_{w',w,\lambda}=v^{l(w')-l(w)}\sum_{i\in\mathbf{Z}}N_{i,w',w,\lambda}v^i$, that is,

$p^\lambda_{z',z}=v^{l(w')-l(w)}\sum_{i\in\mathbf{Z}}N_{i,w',w,\lambda}v^i$ if $w'\mathbf{W}_\lambda=w\mathbf{W}_\lambda$ and z,z' are as in (d),

$N_{i,w',w,\lambda}=0$ if $w'\mathbf{W}_\lambda\ne w\mathbf{W}_\lambda$.

Note that $N_{i,w',w,\lambda}$ is 0 unless i is even.

40.3. Let $B^*\in\mathcal{B}$. Let $U^*=U_{B^*}$ and let T be a maximal torus of B^*. Let $\mathbf{r}=\dim\mathbf{T}$. Let $W_T=N_{G^0}T/T$. We identify $T=\mathbf{T},W_T=\mathbf{W}$ as in 28.5. For any $w\in\mathbf{W}$ we denote by $\dot w$ a representative of w in $N_{G^0}T$.

Let $C=G^0/U^*\times G^0/U^*$. We have a partition $C=\cup_{w\in\mathbf{W}}C_w$ where

$C_w=\{(hU^*,h'U^*)\in C;h^{-1}h'\in B^*\dot wB^*\}$.

For $w\in\mathbf{W}$ let $d_w=\dim C_w$ and let

$\bar C_w=\{(hU^*,h'U^*)\in C;h^{-1}h'\in\overline{B^*\dot wB^*}\}$

(closure in G^0). Now $\bar C_w$ is an irreducible variety and we have a partition $\bar C_w=\cup_{w';w'\le w}C_{w'}$ with C_w smooth, open dense in $\bar C_w$.

Define $\gamma_{\dot w}:B^*\dot wB^*\to T$ by $\gamma_{\dot w}(g)=t$ where $g\in U^*\dot wtU^*$ with $t\in T$. Define $\psi:C_w\to T$ by $\psi(hU^*,h'U^*)=\gamma_{\dot w}(h^{-1}h')$.

For $\mathcal{L}\in\mathfrak{s}$ we set $\mathcal{L}_w=\psi^*\mathcal{L}$, a local system on C_w. (Using 28.1(c) we see that the isomorphism class of $\psi^*\mathcal{L}$ is independent of the choice of $\dot w$.) Let $\mathcal{L}^\sharp_w=IC(\bar C_w,\mathcal{L}_w)\in\mathcal{D}(\bar C_w)$.

40.4. For $w\in\mathbf{W},\mathcal{L}\in\mathfrak{s}$ let $\underline{\mathcal{L}}_w=j_{w!}\mathcal{L}_w$, $\underline{\mathcal{L}}^\sharp_w=\bar j_{w!}\mathcal{L}^\sharp_w$ where $j_w:C_w\to C$, $\bar j_w:\bar C_w\to C$ are the inclusions. Let $\hat C$ be the full subcategory of $\mathcal{D}(C)$ whose objects are the simple perverse sheaves on C which are equivariant for the $G^0\times T\times T$ action

(a) $(x,t,t'):(hU^*,h'U^*)\mapsto(xht^nU^*,xh't'^nU^*)$

on C (for some $n\in\mathbf{N}^*_{\mathbf{k}}$) or equivalently, are isomorphic to $\underline{\mathcal{L}}^\sharp_w[d_w]$ for some $\mathcal{L}\in\mathfrak{s}$ and some $w\in\mathbf{W}$. Let $\mathcal{D}^{cs}(C)$ be the subcategory of $\mathcal{D}(C)$ whose objects are those $K\in\mathcal{D}(C)$ such that for any j, any simple subquotient of ${}^pH^jK$ is in $\hat C$.

If $w,\mathcal{L}$ are as above then $\underline{\mathcal{L}}_w\in\mathcal{D}^{cs}(C)$. Indeed this constructible sheaf is equivariant for the action (a) (for some n) hence so is each ${}^pH^j(\underline{\mathcal{L}}_w)$.

We have a diagram $C\times C\xleftarrow{r}(G^0/U^*)^3\xrightarrow{s}C$ where

$r(h_1U^*,h_2U^*,h_3U^*)=((h_1U^*,h_2U^*),(h_2U^*,h_3U^*))$,

$s(h_1U^*,h_2U^*,h_3U^*)=(h_1U^*,h_3U^*)$.

We define a bi-functor $\mathcal{D}(C)\times\mathcal{D}(C)\to\mathcal{D}(C)$ by $A,A'\mapsto A*A'=s_!r^*(A\boxtimes A')$. The "product" $A*A'$ is associative in an obvious sense. We show:

(b) $A,A'\mapsto A*A'$ restricts to a bi-functor $\mathcal{D}^{cs}(C)\times\mathcal{D}^{cs}(C)\to\mathcal{D}^{cs}(C)$.

Let $A,A'\in\mathcal{D}^{cs}(C)$. To show that $A*A'\in\mathcal{D}^{cs}(C)$ we may assume that $A,A'\in\hat C$.

Then each ${}^pH^j(A*A')$ is equivariant for the action (a) (for some n). This proves (b).

40.5. For $w'\le w$ in $\mathbf{W}$, $\lambda\in\underline{\mathfrak{s}}$, $\mathcal{L}\in\lambda$ and $i\in\mathbf{Z}$ we show:

(a) $\mathcal{H}^i(\mathcal{L}_w^\sharp)|_{C_{w'}}\cong(\mathcal{L}_{w'}(-i/2))^{\oplus N_{i,w',w,\lambda}}$.

(Both sides are 0 unless i is even.)

Let

$\tilde C_w=\{(h,h')\in G^0\times G^0; h^{-1}h'\in B^*\dot wB^*\}\times\mathbf{k}^*$,

$\bar{\tilde C}_w=\{(h,h')\in G^0\times G^0; h^{-1}h'\in\overline{B^*\dot wB^*}\}\times\mathbf{k}^*$.

Now $\bar{\tilde C}_w$ is an irreducible variety and we have a partition $\bar{\tilde C}_w=\cup_{w';w'\le w}\tilde C_{w'}$ with $\tilde C_w$ smooth, open dense in $\bar{\tilde C}_w$. Define $\bar d:\bar{\tilde C}_w\to\bar C_w$, $d:\tilde C_w\to\bar C_w$ by $(h,h',z)\mapsto(hU^*,h'U^*)$. Let $\tilde{\mathcal{L}}_w=d^*\mathcal{L}_w$, a local system on $\tilde C_w$. Let $\tilde{\mathcal{L}}_w^\sharp=IC(\bar{\tilde C}_w,\tilde{\mathcal{L}}_w)$. Since $d,\bar d$ are principal $U^*\times\mathbf{k}^*$-bundles it is enough to show

(b) $\mathcal{H}^i(\tilde{\mathcal{L}}_w^\sharp)|_{\tilde C_{w'}}\cong(\tilde{\mathcal{L}}_{w'}(-i/2))^{\oplus N_{i,w',w,\lambda}}$.

(Both sides are 0 unless i is even.)

We choose $\kappa\in\mathrm{Hom}(T,\mathbf{k}^*),\mathcal{E}\in\mathfrak{s}(\mathbf{k}^*)$ such that $\mathcal{L}\cong\kappa^*\mathcal{E}$, see 28.1(c).

Now B^* acts on $(B^*\dot wB^*)\times\mathbf{k}^*$ and on $(\overline{B^*\dot wB^*})\times\mathbf{k}^*$ by $t_1u:(g,z)\mapsto(g(t_1u)^{-1},\kappa(t_1)z)$ where $t_1\in T$, $u\in U^*$. Let $\bar{\mathbf{P}}_w^\kappa=((\overline{B^*\dot wB^*})\times\mathbf{k}^*)/B^*$, $PP_w^\kappa=((B^*\dot wB^*)\times\mathbf{k}^*)/B^*$. Now $\mathbf{P}_w^\kappa$ is a smooth open dense subvariety of the irreducible variety $\bar{\mathbf{P}}_w^k$ and $\bar{\mathbf{P}}_w^\kappa=\cup_{w';w'\le w}\mathbf{P}_{w'}^\kappa$ is a partition. The morphism $(B^*\dot wB^*)\times\mathbf{k}^*\to\mathbf{k}^*$ given by $(g,z)\mapsto\kappa(\gamma_{\dot w}(g))z$ factors through a morphism $\phi:\mathbf{P}_w^\kappa\to\mathbf{k}^*$. Let $\mathcal{E}_w^\kappa=\phi^*\mathcal{E}$, a local system of rank 1 on $\mathbf{P}_w^\kappa$. Let $\mathcal{E}_w^{\kappa\sharp}=IC(\bar{\mathbf{P}}_w^\kappa,\mathcal{E}_w^\kappa)\in\mathcal{D}(\bar{\mathbf{P}}_w^\kappa)$. From [L14, 1.24] we see that

(c) $\mathcal{H}^i(\mathcal{E}_w^{\kappa\sharp})|_{\mathbf{P}_{w'}^\kappa}\cong(\mathcal{E}_{w'}^\kappa(-i/2))^{\oplus N_{i,w',w,\lambda}}$.

(Both sides are 0 unless i is even.)

We can find $n\in\mathbf{N}_\mathbf{k}^*$ such that $\mathcal{E}\in\mathfrak{s}_n(\mathbf{k}^*)$. Define $\bar c:\bar{\tilde C}_w\to\bar{\mathbf{P}}_w$, $c:\bar{\tilde C}_w\to\bar{\mathbf{P}}_w$ by $(h,h',z)\mapsto B^*-$orbit of $(h^{-1}h',z^n)$. Now $\bar c,c$ are locally trivial fibrations with smooth fibres of pure dimension. Hence (b) follows from (c) provided that we can show that $c^*\mathcal{E}_{w'}^\kappa=\tilde{\mathcal{L}}_{w'}$ for $w'\le w$. We may assume that $w=w'$. We have a commutative diagram

$$\begin{tikzcd}
\mathbf{P}_w^\kappa \arrow[d, "\phi"'] & \tilde C_w\times\mathbf{k}^* \arrow[l, "c"'] \arrow[r, "d"] \arrow[d, "\phi'"'] & C_w \arrow[d, "\kappa\psi"'] \\
\mathbf{k}^* & \mathbf{k}^*\times\mathbf{k}^* \arrow[l, "c'"'] \arrow[r, "d'"] & \mathbf{k}^*
\end{tikzcd}$$

with ϕ,ψ,c,d as above, $\phi'(h,h',z)=(\kappa(\gamma_{\dot w}(h^{-1}h')),z)$, $c'(z',z)=z'z^n$, $d'(z',z)=z'$. Using this and the definitions we have $\tilde{\mathcal{L}}_w=\phi'^*d'^*\mathcal{E}$, $c^*\mathcal{E}_w=\phi'^*c'^*\mathcal{E}$. It remains to show that $d'^*\mathcal{E}=c'^*\mathcal{E}$. This expresses the fact that $\mathcal{E}$ is equivariant for the $\mathbf{k}^*$-action $z_1:z\mapsto z_1^nz$ on $\mathbf{k}^*$ which follows from $\mathcal{E}\in\mathfrak{s}_n(\mathbf{k}^*)$. This proves (b) hence (a).

40.6. Let $w, w' \in \mathbf{W}$, $\mathcal{L}, \mathcal{L}' \in \mathfrak{s}$. We set $L = \underline{\mathcal{L}}_w * \underline{\mathcal{L}}'_{w'} \in \mathcal{D}^{cs}(C)$. Let

$$X = \{(h_1U^*, h_2U^*, h_3U^*) \in (G^0/U^*)^3; h_1^{-1}h_2 \in B^*\dot{w}B^*, h_2^{-1}h_3 \in B^*\dot{w}'B^*\},$$

$$\begin{aligned}\bar{X} = \{&(h_1U^*, h_2B^*, h_3U^*) \in G^0/U^* \times G^0/B^* \times G^0/U^*;\\ &h_1^{-1}h_2 \in B^*\dot{w}B^*, h_2^{-1}h_3 \in B^*\dot{w}'B^*\}.\end{aligned}$$

We have a commutative diagram with a cartesian square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\tau"'] & \bar{X} \arrow[r, "\bar{\sigma}"] \arrow[d, "\bar{\tau}"'] & C \\ T \times T \arrow[r, "f'"] & T & \end{tikzcd}$$

where f is given by $(h_1U^*, h_2U^*, h_3U^*) \mapsto (h_1U^*, h_2B^*, h_3U^*)$,
f' is $(t, t') \mapsto \mathrm{Ad}(\dot{w}')^{-1}(t)t'$,
τ is $(h_1U^*, h_2U^*, h_3U^*) \mapsto (t, t')$ with $h_1^{-1}h_2 \in U^*\dot{w}tU^*, h_2^{-1}h_3 \in U^*\dot{w}'t'U^*$,
$\bar{\tau}$ is $(h_1U^*, h_2B^*, h_3U^*) \mapsto \mathrm{Ad}(\dot{w}')^{-1}(t)t'$ with t, t' as in the definition of τ,
$\bar{\sigma}$ is $(h_1U^*, h_2B^*, h_3U^*) \mapsto (h_1U^*, h_3U^*)$.
From the definitions we have $L = \bar{\sigma}_! f_! \tau^*(\mathcal{L} \boxtimes \mathcal{L}')$. Using the diagram above, we have $L = \bar{\sigma}_! \bar{\tau}^* f'_!(\mathcal{L} \boxtimes \mathcal{L}')$. From the definitions we see that either (i) or (ii) below holds:
(i) $\mathcal{L} \not\cong (\mathrm{Ad}(\dot{w}')^{-1})^*\mathcal{L}'$ and $f'_!(\mathcal{L} \boxtimes \mathcal{L}') = 0$;
(ii) $\mathcal{L} \cong (\mathrm{Ad}(\dot{w}')^{-1})^*\mathcal{L}'$ and $\mathcal{L} \boxtimes \mathcal{L}' = f'^*\mathcal{L}'$.
If (i) holds then $K = 0$. If (ii) holds then, as in 32.16, we have

$$f'_!(\mathcal{L} \boxtimes \mathcal{L}') = f'_! f'^* \mathcal{L}' = \mathcal{L}' \otimes f'_! \bar{\mathbf{Q}}_l \approx \{\mathcal{L}' \otimes \mathcal{H}^e(f'_! \bar{\mathbf{Q}}_l)[-e], e \in \mathbf{Z}\},$$

$$\mathcal{L}' \otimes \mathcal{H}^e(f'_! \bar{\mathbf{Q}}_l)[-e] \approx \{\mathcal{L}'(\mathbf{r} - e), \dots, \mathcal{L}'(\mathbf{r} - e), (\binom{\mathbf{r}}{2\mathbf{r} - e} \text{ copies})\}.$$

Setting $\bar{L} = \bar{\sigma}_! \bar{\tau}^*(\mathcal{L}')$, it follows that

$$L \approx \{\bar{L}(\mathbf{r} - e)[-e], \dots, \bar{L}(\mathbf{r} - e)[-e], (\binom{\mathbf{r}}{2\mathbf{r} - e} \text{ copies}), e \in \mathbf{Z}\}.$$

We now consider $\bar{L}$ for certain choices of w, w'.

If w, w' satisfy $l(ww') = l(w) + l(w')$ then $\bar{\sigma}$ restricts to an isomorphism $\bar{X} \to C_{ww'}$ and $\bar{L} = \underline{\mathcal{L}}'_{ww'}$.

Now assume that $\alpha, \check{\alpha}, s_\alpha$ are as in 28.3 and that $w = w' = s_\alpha \in \mathbf{I}$. We have
$\bar{L} \approx \{j_{u!}\bar{L}_u; u \in \mathbf{W}\}$
where $j_u : C_u \to C$ is the inclusion and $\bar{L}_u = j_u^* \bar{L}$. Let $\bar{X}_u = \bar{\sigma}^{-1}(C_u)$. Then $\bar{L}_u = \bar{\sigma}_{u!} \bar{\tau}_u^*(\mathcal{L}')$ where $\bar{\sigma}_u : \bar{X}_u \to C_u$, $\bar{\tau}_u : \bar{X}_u \to T$ are the restrictions of $\bar{\sigma}, \bar{\tau}$.

If $u \notin \{1, s_\alpha\}$ then $\bar{X}_u = \emptyset$ and $\bar{L}_u = 0$. If $u = 1$ then $\bar{\sigma}_u : \bar{X}_u \to C_u$ is an affine line bundle and $\bar{\tau}_u^*(\mathcal{L}') = \bar{\sigma}_u^*\mathcal{L}'_u$; hence $\bar{\sigma}_{u!}\bar{\tau}_u^*(\mathcal{L}') = \bar{\sigma}_{u!}\bar{\sigma}_u^*\mathcal{L}'_u = \mathcal{L}'_u[[-1]]$. If $u = s_\alpha$ then $\bar{\sigma}_u : \bar{X}_u \to C_u$ is a principal $\mathbf{k}^*$-bundle and either (iii) or (iv) below holds:

(iii) $\check{\alpha}^*\mathcal{L}' \ncong \bar{\mathbf{Q}}_l$ and $\bar{\sigma}_{u!}\bar{\tau}_u^*(\mathcal{L}') = 0$,

(iv) $\check{\alpha}^*\mathcal{L}' \cong \bar{\mathbf{Q}}_l$ and $\bar{\tau}_u^*(\mathcal{L}') = \bar{\sigma}_u^*\mathcal{L}'_u$.

If (iv) holds then, as in case (ii) above, we have

$\bar{\sigma}_{u!}\bar{\tau}_u^*(\mathcal{L}') = \bar{\sigma}_{u!}\bar{\sigma}_u^*\mathcal{L}'_u = \mathcal{L}'_u \otimes \bar{\sigma}_{u!}\bar{\mathbf{Q}}_l \Leftrightarrow \{\mathcal{L}'_u \otimes \mathcal{H}^e(\bar{\sigma}_{u!}\bar{\mathbf{Q}}_l)[-e], e \in \mathbf{Z}\}$,

$\mathcal{L}'_u \otimes \mathcal{H}^e(\bar{\sigma}_{u!}\bar{\mathbf{Q}}_l)[-e] \Leftrightarrow \{\mathcal{L}'_u(1-e), \dots, \mathcal{L}'_u(1-e), (\binom{1}{2-e}$ copies$)\}$.

40.7. In this subsection we assume that $\mathbf{k}$ is an algebraic closure of a finite field. Now the $\mathcal{A}$-module $\mathfrak{K}(C)$ is defined as in 36.8 (the character sheaves on C are taken to be the objects in $\hat{C}$).

For $(w, \lambda) \in \mathbf{W} \times \underline{\mathfrak{s}}$, let $[w; \lambda]$ be the basis element of $\mathfrak{K}(C)$ given by $\underline{\mathcal{L}}_w^\sharp[[d_w/2]]$; we choose $\mathcal{L} \in \lambda$ and we regard $\underline{\mathcal{L}}_w, \underline{\mathcal{L}}_w^\sharp$ as mixed complexes on C whose restriction to C_w is pure of weight 0; then $gr(\underline{\mathcal{L}}_w), gr(\underline{\mathcal{L}}_w^\sharp)$ are defined in $\mathfrak{K}(C)$ as in 36.8. We denote these elements of $\mathfrak{K}(C)$ by $[w; \lambda]', [w; \lambda]'^\sharp$ respectively. From 40.5(a) we see that

(a) $(-v)^{d_w}[w; \lambda] = [w; \lambda]'^\sharp = \sum_{w' \in \mathbf{W}} \sum_{i \in 2\mathbf{Z}} N_{i,w',w,\lambda} v^i [w'; \lambda]'$ in $\mathfrak{K}(C)$.

where $N_{i,w',w,\lambda}$ is as in 40.2(e).

Let r, s be as in 40.4. By 40.4(b), $s_! r^* : \mathcal{D}(C \times C) \to \mathcal{D}(C)$ restricts to a functor $\mathcal{D}^{cs}(C \times C) \to \mathcal{D}^{cs}(C)$ where the character sheaves on $C \times C$ are by definition complexes of the form $A \boxtimes A'$ with $A \in \hat{C}, A' \in \hat{C}$. Hence the $\mathcal{A}$-linear map $gr(s_! r^*) : \mathfrak{K}(C \times C) \to \mathfrak{K}(C)$ or equivalently $\mathfrak{K}(C) \otimes_{\mathcal{A}} \mathfrak{K}(C) \to \mathfrak{K}(C)$ is well defined. (We have canonically $\mathfrak{K}(C \times C) = \mathfrak{K}(C) \otimes_{\mathcal{A}} \mathfrak{K}(C)$.) We write $\xi * \xi'$ instead of $gr(s_! r^*)(\xi \boxtimes \xi')$ where $\xi, \xi' \in \mathfrak{K}(C)$. Note that $\xi, \xi' \mapsto \xi * \xi'$ defines an associative $\mathcal{A}$-algebra structure on $\mathfrak{K}(C)$.

Let $w, w' \in \mathbf{W}$, $\lambda, \lambda' \in \underline{\mathfrak{s}}$. From 40.6 we see that:

if $w'\lambda' \ne \lambda$ *then* $[w; \lambda]' * [w'; \lambda']' = 0$ *in* $\mathfrak{K}(C)$*;*

if $w'\lambda' = \lambda$ *and* $l(ww') = l(w) + l(w')$ *then* $[w; \lambda]' * [w', \lambda']' = (v^2 - 1)^{\mathbf{r}}[ww'; \lambda']'$ *in* $\mathfrak{K}(C)$*;*

if $s \in \mathbf{I}$ *and* $s\lambda' = \lambda$ *then* $[s; \lambda]' * [s, \lambda']' = (v^2 - 1)^{\mathbf{r}}(v^2[1; \lambda']' + (v^2 - 1)c[s; \lambda']')$ *where* $c = 1$ *for* $s \in \mathbf{W}_{\lambda'}$ *and* $c = 0$ *for* $s \notin \mathbf{W}_{\lambda'}$*.*

Using this and (a), 40.1(a), 40.2(e), we see that

(b) *the unique* $\mathcal{A}$*-linear isomorphism* $\omega : \mathfrak{K}(C) \to H$ *(*H *as in 40.1) given by* $[w, \lambda]' \mapsto v^{l(w)}\tilde{T}_w 1_\lambda$ *for* $w \in \mathbf{W}$*,* $\lambda \in \underline{\mathfrak{s}}$*, satisfies* $\omega([w, \lambda]) = (-v)^{-d_w} v^{l(w)} c_{w,\lambda}$ *for* $w \in \mathbf{W}$*,* $\lambda \in \underline{\mathfrak{s}}$ *and* $\omega(x * x') = (v^2 - 1)^{\mathbf{r}}\omega(x)\omega(x')$ *for any* $x, x' \in \mathfrak{K}(C)$*.*

40.8. For $w, w' \in \mathbf{W}$ and $\lambda, \lambda' \in \underline{\mathfrak{s}}$ we have

$c_{w,\lambda} c_{w',\lambda'} = \sum_{y \in \mathbf{W}, \nu \in \underline{\mathfrak{s}}} \gamma_{y,\nu}^{w,\lambda;w',\lambda'} c_{y,\lambda}$

in the algebra H. Here $\gamma_{y,\nu}^{w,\lambda;w',\lambda'} \in \mathcal{A}$. We have:

(a) $\gamma_{y,\nu}^{w,\lambda;w',\lambda'} \in \mathbf{N}[v, v^{-1}]$.

By the arguments in 34.4-34.10 (with $D = G^0$) this is reduced to the analogous

(well known) statement for the structure constants of the algebra H^D_λ with its basis (c^λ_w) (see 34.2).

40.9. For any $J \subset \mathbf{I}$ let H_J be the $\mathcal{A}$-submodule of H spanned by $\{c_{w,\lambda}; w \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}\}$ or equivalently by $\{\tilde{T}_w 1_\lambda; w \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}\}$. From the definitions we see that H_J is a subalgebra of H. For any $J \subset \mathbf{I}, J' \subset \mathbf{I}$ we define a relation $\preceq_{J,J'}$ on $\mathbf{W} \times \underline{\mathfrak{s}}$ as follows. We say that $(y,\nu) \preceq_{J,J'} (w,\lambda)$ if there exist $w_1 \in \mathbf{W}_J, w_2 \in \mathbf{W}_{J'}$, $\lambda_1, \lambda_2 \in \underline{\mathfrak{s}}$ such that in the expansion (in the algebra H):
$c_{w_1,\lambda_1} c_{w,\lambda} c_{w_2,\lambda_2} = \sum_{y' \in \mathbf{W}, \nu' \in \underline{\mathfrak{s}}} a_{y',\nu'} c_{y',\nu'}$
(with $a_{y',\nu'} \in \mathcal{A}$) we have $a_{y,\nu} \neq 0$.

Using the associativity of the product in H, the fact that $H_J, H_{J'}$ are subalgebras of H and 40.8(a), we see that $\preceq_{J,J'}$ is transitive. Using the formula $c_{1,w\lambda} c_{w,\lambda} c_{1,\lambda} = c_{w,\lambda}$ we see that it is reflexive. Thus, it is a preorder. Let $\sim_{J,J'}$ be the equivalence relation attached to $\preceq_{J,J'}$; thus, $(y,\nu) \sim_{J,J'} (w,\lambda)$ if $(y,\nu) \preceq_{J,J'} (w,\lambda)$ and $(w,\lambda) \preceq_{J,J'} (y,\nu)$. The equivalence classes for $\sim_{J,J'}$ are called (J,J')-two-sided cells. The $(\mathbf{I},\mathbf{I})$-two sided cells in $\mathbf{W} \times \underline{\mathfrak{s}}$ are also called two-sided cells.

40.10. Let $w, w', w'' \in \mathbf{W}$, $\mathcal{L}, \mathcal{L}', \mathcal{L}'' \in \mathfrak{s}$. We set $K = \underline{\mathcal{L}}_w * \underline{\mathcal{L}'}_{w'}^{\sharp} * \underline{\mathcal{L}}''_{w''} \in \mathcal{D}^{cs}(C)$. Let

$$X = \{(h_1U^*, h_2U^*, h_3U^*, h_4U^*) \in (G^0/U^*)^4;$$
$$h_1^{-1}h_2 \in B^*\dot{w}B^*, h_2^{-1}h_3 \in \overline{B^*\dot{w}'B^*}, h_3^{-1}h_4 \in B^*\dot{w}''B^*\},$$

an irreducible variety. Let X_0 be the smooth open dense subset of X defined by the condition $h_2^{-1}h_3 \in B^*\dot{w}'B^*$. Define $\sigma : X \longrightarrow C$ by
$(h_1U^*, h_2U^*, h_3U^*, h_4U^*) \mapsto (h_1U^*, h_4U^*)$.
Define $\tau : X_0 \longrightarrow T \times T \times T$ by
$(h_1U^*, h_2U^*, h_3U^*, h_4U^*) \mapsto (t, t', t'')$
with $h_1^{-1}h_2 \in U^*\dot{w}tU^*, h_2^{-1}h_3 \in U^*\dot{w}'t'U^*, h_3^{-1}h_4 \in U^*\dot{w}''t''U^*$.
Let $\mathcal{F} = \tau^*(\mathcal{L} \boxtimes \mathcal{L}' \boxtimes \mathcal{L}'')$, a local system on X_0. Then $\mathcal{F}^\sharp := IC(X, \mathcal{F}) \in \mathcal{D}(X)$ is defined and we have $K = \sigma_! \mathcal{F}^\sharp$.

Let $\bar{X}$ (resp. $\bar{X}_0$) be the the variety of all $(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \in G^0/U^* \times G^0/B^* \times G^0/B^* \times G^0/U^*$ that satisfy the same equations as those defining X (resp. X_0). Note that $\bar{X}$ is irreducible and $\bar{X}_0$ is an open dense smooth subset of $\bar{X}$. We have a cartesian diagram

$$\begin{array}{ccccc}
X & \xrightarrow{f} & \bar{X} & \xrightarrow{\bar{\sigma}} & C \\
\uparrow & & \uparrow & & \\
X_0 & \xrightarrow{f_0} & \bar{X}_0 & & \\
{\scriptstyle\tau}\downarrow & & {\scriptstyle\bar{\tau}}\downarrow & & \\
T \times T \times T & \xrightarrow{f'} & T & &
\end{array}$$

where $X_0 \to X, \bar{X}_0 \to \bar{X}$ are the obvious imbeddings,

f, f_0 are given by $(h_1U^*, h_2U^*, h_3U^*, h_4U^*) \mapsto (h_1U^*, h_2B^*, h_3B^*, h_4U^*)$,

f' is $(t, t', t'') \mapsto \mathrm{Ad}(\dot{w}'\dot{w}'')^{-1}(t)\mathrm{Ad}(\dot{w}'')^{-1}(t')t''$,

$\bar{\tau}$ is $(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \mapsto \mathrm{Ad}(\dot{w}'\dot{w}'')^{-1}(t)\mathrm{Ad}(\dot{w}'')^{-1}(t')t''$ with t, t', t'' as in the definition of τ,

$\bar{\sigma}$ is $(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \mapsto (h_1U^*, h_4U^*)$.

Assume that $\mathcal{L} \cong (\mathrm{Ad}(\dot{w}')^{-1})^*\mathcal{L}'$ and $\mathcal{L}' \cong (\mathrm{Ad}(\dot{w}'')^{-1})^*\mathcal{L}''$. Then $\mathcal{L} \boxtimes \mathcal{L}' \boxtimes \mathcal{L}'' = f'^*\mathcal{L}''$. We have $\mathcal{F} = \tau^* f'^*\mathcal{L}'' = f_0^*\bar{\tau}^*\mathcal{L}''$. Since f is a principal $T \times T$-bundle and $X_0 = f^{-1}(\bar{X}_0)$ it follows that $\mathcal{F}^\sharp = f^*IC(\bar{X}, \bar{\tau}^*\mathcal{L}'')$. Note that $f_!\bar{\mathbf{Q}}_l \rightsquigarrow \{\mathcal{H}^e(f_!\bar{\mathbf{Q}}_l)[-e], 2\mathbf{r} \le e \le 4\mathbf{r}\}$,

$$\mathcal{H}^e(f_!\bar{\mathbf{Q}}_l) \rightsquigarrow \{\bar{\mathbf{Q}}_l(2\mathbf{r}-e), \dots, \bar{\mathbf{Q}}_l(2\mathbf{r}-e), (\binom{2\mathbf{r}}{4\mathbf{r}-e} \text{ copies})\}.$$

Hence setting $\bar{K} = \bar{\sigma}_!(IC(\bar{X}, \bar{\tau}^*\mathcal{L}''))$ we have

$$K = \sigma_! f^*IC(\bar{X}, \bar{\tau}^*\mathcal{L}'') = \bar{\sigma}_! f_! f^*IC(\bar{X}, \bar{\tau}^*\mathcal{L}'') = \bar{\sigma}_!(IC(\bar{X}, \bar{\tau}^*\mathcal{L}'') \otimes f_!\bar{\mathbf{Q}}_l),$$

(a) $\quad K \rightsquigarrow \{\bar{K}(2\mathbf{r}-e)[-e], \dots, \bar{K}(2\mathbf{r}-e)[-e], (\binom{2\mathbf{r}}{4\mathbf{r}-e} \text{ copies}), 2\mathbf{r} \le e \le 4\mathbf{r}\}.$

We now show:

(b) *if $A \in \hat{C}$ is such that $A \dashv \bar{K}$, then $A \dashv K$.*

We may regard $\mathcal{L}, \mathcal{L}', \mathcal{L}''$ as mixed local systems (with respect to a rational structure over a sufficiently large finite subfield of $\mathbf{k}$) which are pure of weight 0. Then $K, \bar{K}$ are naturally mixed complexes and (a) is compatible with the mixed structures. For any mixed perverse sheaf P, let P_h be the subquotient of P of pure weight h. We can find $h \in \mathbf{Z}$ such that $A \dashv {}^pH^j(\bar{K})_h$ for some $j \in \mathbf{Z}$; moreover we may assume that h is maximum possible. Note that $A \dashv {}^pH^{j+4\mathbf{r}}(\bar{K}[-4\mathbf{r}](-2\mathbf{r}))_{h+2\mathbf{r}}$ and $A \not\dashv {}^pH^{j'}(\bar{K}[-e](2\mathbf{r}-e))_{h+2\mathbf{r}}$ for $2\mathbf{r} \le e < 4\mathbf{r}$ and any j'; hence from (a) we see that $A \dashv {}^pH^{j+4\mathbf{r}}(K)_{h+2\mathbf{r}}$. In particular, $A \dashv K$, and (b) is proved.

40.11. Let $w, w'\mathcal{L}, \mathcal{L}', X, \bar{X}, \tau$ be as in 40.6. We set $\mathbf{L} = \underline{\mathcal{L}}_w^\sharp * \underline{\mathcal{L}}'_{w'}{}^\sharp \in \mathcal{D}^{cs}(C)$. Let $A = \underline{\mathcal{L}}''_{w''}{}^\sharp[d_{w''}]$. We show:

(a) *If $A \dashv \mathbf{L}$ then $[w'', \lambda'']$ appears with non-zero coefficient in the expansion of the product $[w, \lambda] * [w', \lambda']$ in terms of the basis $([y, \nu])$ of $\mathfrak{K}(C)$.*

Let

$$\mathbf{X} = \{(h_1U^*, h_2U^*, h_3U^*) \in (G^0/U^*)^3; h_1^{-1}h_2 \in \overline{B^*\dot{w}B^*}, h_2^{-1}h_3 \in \overline{B^*\dot{w}'B^*}\},$$

$$\bar{\mathbf{X}} = \{(h_1U^*, h_2B^*, h_3U^*) \in G^0/U^* \times G^0/B^* \times G^0/U^*;$$
$$h_1^{-1}h_2 \in \overline{B^*\dot{w}B^*}, h_2^{-1}h_3 \in \overline{B^*\dot{w}'B^*}\}.$$

Note that X (resp. $\bar{X}$) is naturally an open dense subset of $\mathbf{X}$ (resp. $\bar{\mathbf{X}}$). Define $\sigma' : \mathbf{X} \to C$ by $(h_1U^*, h_2U^*, h_3U^*) \mapsto (h_1U^*, h_3U^*)$. Define $\bar{\sigma}' : \bar{\mathbf{X}} \to C$ by $(h_1U^*, h_2B^*, h_3U^*) \mapsto (h_1U^*, h_3U^*)$. Let $\mathcal{F} = \tau^*(\mathcal{L} \boxtimes \mathcal{L}')$, a local system on X. Then $\mathcal{F}^\sharp := IC(\mathbf{X}, \mathcal{F}) \in \mathcal{D}(\mathbf{X})$ is defined and we have $\mathbf{L} = \sigma'_!\mathcal{F}^\sharp$. We have a cartesian diagram

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "\tilde{f}"] & \bar{\mathbf{X}} \arrow[r, "\bar{\sigma}'"] & C \\
X \arrow[u] \arrow[r, "f"] \arrow[d, "\tau"'] & \bar{X} \arrow[u] \arrow[d, "\bar{\tau}"'] & \\
T \times T \arrow[r, "f'"] & T &
\end{tikzcd}$$

where $X \to \mathbf{X}, \bar{X} \to \bar{\mathbf{X}}$ are the obvious imbeddings, $f, f', \bar{\tau}$ are as in 40.6 and $\tilde{f}$ is the obvious map.

Assume first that 40.6(i) holds. Let $m' : T \times \mathbf{X} \to \mathbf{X}$ be the free T-action $t_1 : (h_1U^*, h_2U^*, h_3U^*) \mapsto (h_1U^*, h_2t_1^{-1}U^*, h_3U^*)$. This restricts to a free T-action $m : T \times X \to X$. Define a free T action $m_0 : T \times (T \times T) \to T \times T$ by $t_1 : (t, t') \mapsto (t_1^{-1}t, \mathrm{Ad}(\dot{w}')^{-1}(t_1)t'$. Then m, m_0 are compatible with τ. By our assumption we have $m_0^*(\mathcal{L} \boxtimes \mathcal{L}') = \mathcal{L}_0 \boxtimes \mathcal{L} \boxtimes \mathcal{L}'$ where $\mathcal{L}_0 \in \mathfrak{s}(T)$, $\mathcal{L}_0 \not\cong \bar{\mathbf{Q}}_l$. It follows that $m^*(\mathcal{F}) \cong \mathcal{L}_0 \boxtimes \mathcal{F}$. From the properties of intersection cohomology we then have $m'^*(\mathcal{F}^\sharp) \cong \mathcal{L}_0 \boxtimes \mathcal{F}^\sharp$. Let $r : T \times \mathbf{X} \to \mathbf{X}$ be the second projection. Since $\mathcal{L}_0 \in \mathfrak{s}(T)$, $\mathcal{L}_0 \not\cong \bar{\mathbf{Q}}_l$, we have $r_!(\mathcal{L}_0 \boxtimes \mathcal{F}^\sharp) = 0$. Hence $r_!m'^*(\mathcal{F}^\sharp) = 0$. Since m', f', r, f' form a cartesian diagram we must have $f'^*f'(\mathcal{F}^\sharp) = 0$. Since f' is a principal T-bundle we deduce that $f'_!(\mathcal{F}^\sharp) = 0$. We have $\mathbf{L} = \bar{\sigma}'_!f'_!(\mathcal{F}^\sharp)$ hence $\mathbf{L} = 0$. In this case (a) is clear.

Assume next that 40.6(ii) holds. Then $\mathcal{L} \boxtimes \mathcal{L}' = f'^*\mathcal{L}'$ and $\mathcal{F} = \tau^*f'^*\mathcal{L}' = f^*\bar{\tau}^*\mathcal{L}'$. Since f' is a principal T-bundle and $X = f'^{-1}(\bar{X})$ it follows that $\mathcal{F}^\sharp = f'^*IC(\bar{\mathbf{X}}, \bar{\tau}^*\mathcal{L}')$. Note that $f'_!\bar{\mathbf{Q}}_l \approx \{\mathcal{H}^e(f'_!\bar{\mathbf{Q}}_l)[-e], \mathbf{r} \le e \le 2\mathbf{r}\}$,

$$\mathcal{H}^e(f'_!\bar{\mathbf{Q}}_l) \approx \{\bar{\mathbf{Q}}_l(\mathbf{r} - e), \dots, \bar{\mathbf{Q}}_l(\mathbf{r} - e), (\binom{\mathbf{r}}{2\mathbf{r} - e} \text{ copies})\}.$$

Hence setting $\bar{\mathbf{L}} = \bar{\sigma}'_!(IC(\bar{\mathbf{X}}, \bar{\tau}^*\mathcal{L}'))$ we have

$$\mathbf{L} = \sigma'_!f'^*IC(\bar{\mathbf{X}}, \bar{\tau}^*\mathcal{L}') = \bar{\sigma}'_!f'_!f'^*IC(\bar{\mathbf{X}}, \bar{\tau}^*\mathcal{L}') = \bar{\sigma}'_!(IC(\bar{\mathbf{X}}, \bar{\tau}^*\mathcal{L}') \otimes f'_!\bar{\mathbf{Q}}_l),$$

$$\mathbf{L} \approx \{\bar{\mathbf{L}}(\mathbf{r} - e)[-e], \dots, \bar{\mathbf{L}}(\mathbf{r} - e)[-e], (\binom{\mathbf{r}}{2\mathbf{r} - e} \text{ copies}), \mathbf{r} \le e \le 2\mathbf{r}\}.$$

Since $A \dashv \mathbf{L}$, this shows that $A \dashv \bar{\mathbf{L}}$. We regard $\mathcal{L}'$ as a pure local system of weight 0. Then $\bar{\mathbf{L}} = \bar{\sigma}'_!(IC(\bar{\mathbf{X}}, \bar{\tau}^*\mathcal{L}'))$ is again pure of weight 0, since $\bar{\sigma}'$ is proper (see [BBD]). Hence the coefficient with which A appears in the expansion of $gr(\bar{\mathbf{L}})$ is a

polynomial in $-v$ with coefficients given by the multiplicities of A in the various ${}^pH^j(\bar{\mathbf{L}})$; in particular, A appears with coefficient $\ne 0$ in $gr(\bar{\mathbf{L}})$. On the other hand the arguments above show that $[w,\lambda]*[w',\lambda']=(v^2-1)^{\mathbf{r}}gr(\bar{\mathbf{L}})$. It follows that A appears with coefficient $\ne 0$ in $[w,\lambda]*[w',\lambda']$. This proves (a).

41. Character sheaves and two-sided cells

41.1. In this section we preserve the notation of 40.3. We fix a connected component D of G and we pick $\delta\in N_DB^*\cap N_DT$. We write ϵ instead of $\epsilon_D:\mathbf{W}\to\mathbf{W}$. For $w\in\mathbf{W}$ we set

$Z^w_{\emptyset,D}=\{(B,B',xU_B)\in Z_{\emptyset,D};\mathrm{pos}(B,B')=w\}$.

(This is the same as ${}^{w^{-1}}Z_{\emptyset,D}$ in 36.2.) Define $\xi_D:C\to Z_{\emptyset,D}$ by $(hU^*,h'U^*)\mapsto(hB^*h^{-1},h'B^*h'^{-1},h'\delta h^{-1}U_{hB^*h^{-1}})$, a principal T-bundle for the free T-action on C given by $t:(hU^*,h'U^*)\to(htU^*,h'(\delta t\delta^{-1})U^*)$.

Since $\xi_D^{-1}(Z^w_{\emptyset,D})=C_w$, ξ_D restricts to a principal T-bundle $\xi_{D,w}:C_w\to Z^w_{\emptyset,D}$. We have a commutative diagram

$$\begin{tikzcd} T \arrow[d, "\zeta"'] & C_w \arrow[l, "\psi"'] \arrow[r, "="] & C_w \arrow[d, "\xi_{D,w}"] \\ \mathbf{d} & G^0/(U^*\cap\dot wU^*\dot w^{-1})\times\mathbf{d} \arrow[l, "pr_2"'] \arrow[u, "j'"] \arrow[r, "j"] & Z^w_{\emptyset,D} \end{tikzcd}$$

where ψ is as in 40.3,

$\mathbf{d}=\dot w\delta T$,

$j(f(U^*\cap\dot wU^*\dot w^{-1}),s)=(fB^*f^{-1},f\dot wB^*\dot w^{-1}f^{-1},fsf^{-1}U_{fB^*f^{-1}})$,

$j'(f(U^*\cap\dot wU^*\dot w^{-1}),s)=(fU^*,fs\delta^{-1}U^*)$,

$\zeta(t)=\dot w\delta(\delta^{-1}t\delta)$.

Note that the lower row in the diagram is as in 36.2(a).

Define $\iota:\mathbf{d}\to T$ by $\iota(\dot w\delta t)=t$ where $t\in T$. If $\mathcal{L}\in\mathfrak{s}$ is such that $\mathrm{Ad}((\dot wd)^{-1})^*\mathcal{L}\cong\mathcal{L}$ then $pr_2^*\iota^*(\mathcal{L})$ is a local system on $G^0/(U^*\cap\dot wU^*\dot w^{-1})\times\mathbf{d}$, equivariant for the T-action $t_0:(f(U^*\cap\dot wU^*\dot w^{-1}),s)=(ft_0^{-1}(U^*\cap\dot wU^*\dot w^{-1}),t_0st_0^{-1})$ on $G^0/(U^*\cap\dot wU^*\dot w^{-1})\times\mathbf{d}$, which makes j a principal T-bundle. It follows that there is a well defined local system $\dot{\mathcal{L}}_w$ (of rank 1) on $Z^w_{\emptyset,D}$ such that $j^*\dot{\mathcal{L}}_w=pr_2^*\iota^*(\mathcal{L})$. We show:

(a) $\xi^*_{D,w}(\dot{\mathcal{L}}_w)=(\mathrm{Ad}(\delta^{-1})^*\mathcal{L})_w$.

Since j' is an isomorphism it is enough to show that $j'^*\xi^*_{D,w}(\dot{\mathcal{L}}_w)=j'^*(\mathrm{Ad}(\delta^{-1})^*\mathcal{L})_w$ or that $j^*\dot{\mathcal{L}}_w=j'^*(\mathrm{Ad}(\delta^{-1})^*\mathcal{L})_w$ or that $pr_2^*\iota^*\mathcal{L}=j'^*\psi^*(\mathrm{Ad}(\delta^{-1})^*)\mathcal{L})$ or that $j'^*\psi^*\zeta^*\iota^*\mathcal{L}=j'^*\psi^*(\mathrm{Ad}(\delta^{-1})^*)\mathcal{L})$. It is enough to show that $\zeta^*\iota^*\mathcal{L}=\mathrm{Ad}(\delta^{-1})^*\mathcal{L}$. This follows from $\mathrm{Ad}(\delta^{-1})=\iota\zeta:T\to T$.

Let $h_w:Z^w_{\emptyset,D}\to Z_{\emptyset,D}$, $\bar h_w:\bar Z^w_{\emptyset,D}\to Z_{\emptyset,D}$ be the inclusions ($\bar Z^w_{\emptyset,D}=\cup_{w';w'\le w}Z^{w'}_{\emptyset,D}$ is the closure of $Z^w_{\emptyset,D}$ in $Z_{\emptyset,D}$.) Let $\underline{\dot{\mathcal{L}}}_w=h_{w!}\dot{\mathcal{L}}_w$, $\underline{\dot{\mathcal{L}}}^\sharp_w=\bar h_{w!}\dot{\mathcal{L}}^\sharp_w$. Using (a) and the fact that ξ_D is a principal T-bundle we deduce

(b) $\xi_D^*(\dot{\underline{\mathcal{L}}}_w) = \underline{(\mathrm{Ad}(\delta^{-1})^*\mathcal{L})}_w$,
(c) $\xi_D^*(\dot{\underline{\mathcal{L}}}_w^\sharp) = \underline{(\mathrm{Ad}(\delta^{-1})^*\mathcal{L})}_w^\sharp$.
Now let D' be another connected component of G. We pick $\delta' \in N_{D'}B^* \cap N_{D'}T$. We have a commutative diagram with a cartesian right square

$$\begin{tikzcd}
C \times C \arrow[d, "\xi_D \times \xi_{D'}"'] & (G^0/U^*)^3 \arrow[l, "r"'] \arrow[r, "s"] \arrow[d, "\xi_0"'] & C \arrow[d, "\xi_{D'D}"'] \\
Z_{\emptyset,D} \times Z_{\emptyset,D'} & Z_0 \arrow[l, "b_1"'] \arrow[r, "b_2"] & Z_{\emptyset,D'D}
\end{tikzcd}$$

where r, s are as in 40.4, Z_0, b_1, b_2 are as in 32.5 (with $J = \emptyset$) and

$$\xi_0(h_1U^*, h_2U^*, h_3U^*)$$
$$= (h_1B^*h_1^{-1}, h_2B^*h_2^{-1}, h_3B^*h_3^{-1}, h_2\delta h_1^{-1}U_{h_1B^*h_1^{-1}}, h_3\delta' h_2^{-1}U_{h_2B^*h_2^{-1}}).$$

Hence, if $A \in \mathcal{D}(Z_{\emptyset,D}), A' \in \mathcal{D}(Z_{\emptyset,D'})$, then $\xi_{D'D}^* b_{2!}b_1^*(A \boxtimes A') = s_!r^*(\xi_D^*A \boxtimes \xi_{D'}^*A')$, or equivalently
(d) $\xi_{D'D}^*(A * A') = (\xi_D^*A) * (\xi_{D'}^*A')$.

41.2. Let $u \in \mathbf{W}$. Let

$$\Upsilon_u = \{(B, B', g(U_B \cap U_{B'});$$
$$B \in \mathcal{B}, B' \in \mathcal{B}, g(U_B \cap U_{B'}) \in D/(U_B \cap U_{B'}), \mathrm{pos}(B, B') = u\}$$

and let $\Phi_u : \mathcal{D}(Z_{\emptyset,D}) \to \mathcal{D}(Z_{\emptyset,D})$ be the composition $\mathfrak{h}_!\mathfrak{j}^*$ where $\mathfrak{j} : \Upsilon_u \to Z_{\emptyset,D}$ is $(B, B', g(U_B \cap U_{B'}) \mapsto (B, gBg^{-1}, gU_B)$ and $\mathfrak{h} : \Upsilon_u \to Z_{\emptyset,D}$ is
$(B, B', g(U_B \cap U_{B'}) \mapsto (B', gB'g^{-1}, gU_{B'})$.
(A special case of definitions in 37.1.) Let

$$\Upsilon' = \{(B', B, \tilde{B}, \tilde{B}', gU_{B'}); B' \in \mathcal{B}, B \in \mathcal{B}, \tilde{B} \in \mathcal{B}, \tilde{B}' \in \mathcal{B},$$
$$gU_{B'} \in D/U_{B'}, \mathrm{pos}(B', B) = u^{-1}, \mathrm{pos}(\tilde{B}, \tilde{B}') = \epsilon(u), gB'g^{-1} = \tilde{B}'\},$$

$$s : \Upsilon_u \to \Upsilon', (B, B', g(U_B \cap U_{B'}) \mapsto (B', B, gBg^{-1}, gB'g^{-1}, gU_{B'}).$$

Note that s is an isomorphism. (We show this only at the level of sets. Define $s' : \Upsilon' \to \Upsilon_u$ by $(B', B, \tilde{B}, \tilde{B}', gU_{B'}) \mapsto (B, B', x(U_B \cap U_{B'}))$ where $x \in D$ is such that $xBx^{-1} = \tilde{B}$, $xU_{B'} = gU_{B'}$. This is well defined and clearly an inverse of s.) It follows that $\mathfrak{h}_!\mathfrak{j}^* = \mathfrak{h}'_!\mathfrak{j}'^*$ where
$\mathfrak{h}' = \mathfrak{h}s' : \Upsilon' \to Z_{\emptyset,D}$ is $(B', B, \tilde{B}, \tilde{B}', gU_{B'}) \mapsto (B', \tilde{B}', gU_{B'})$,
$\mathfrak{j}' = \mathfrak{j}s' : \Upsilon' \to Z_{\emptyset,D}$ is $(B', B, \tilde{B}, \tilde{B}', gU_{B'}) \mapsto (B, \tilde{B}, xU_B)$
and $x \in D$ is such that $xBx^{-1} = \tilde{B}, xU_{B'} = gU_{B'}$ (then $x(U_B \cap U_{B'})$ is well

defined). We have a commutative diagram with a cartesian right square

$$\begin{tikzcd} C \arrow[d, "\xi_D"'] & \tilde C \arrow[l, "\tilde j"'] \arrow[r, "\tilde h"] \arrow[d, "\xi'"'] & C \arrow[d, "\xi_D"'] \\ Z_{\emptyset,D} & \Upsilon'_u \arrow[l, "\mathfrak{j}'"'] \arrow[r, "\mathfrak{h}'"] & Z_{\emptyset,D} \end{tikzcd}$$

where ξ_D is as in 41.1,

$$\begin{aligned}\tilde C =&\{(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \in (G^0/U^*)^4;\\ &h_1^{-1}h_2 \in B^*\dot u^{-1}B^*, h_3^{-1}h_4 \in B^*\delta\dot u\delta^{-1}B^*\},\end{aligned}$$

$\tilde h$ is $(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \mapsto (h_1U^*, h_4U^*)$,
ξ' is

$$\begin{aligned}&(h_1U^*, h_2B^*, h_3B^*, h_4U^*)\\ &\mapsto (h_1B^*h_1^{-1}, h_2B^*h_2^{-1}, h_3B^*h_3^{-1}, h_4B^*h_4^{-1}, h_4\delta h_1^{-1}U_{h_1B^*h_1^{-1}}),\end{aligned}$$

$\tilde j$ is $(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \mapsto (h_2t^{-1}U^*, h_3\tilde tU^*)$
where $t, \tilde t \in T$ are given by $h_1^{-1}h_2 \in U^*\dot u^{-1}tU^*$, $h_3^{-1}h_4 \in U^*\tilde t\delta\dot u\delta^{-1}U^*$.
We see that for $A \in \mathcal{D}(Z_{\emptyset,D})$ we have

$$\xi_D^*\Phi_u(A) = \xi_D\mathfrak{h}_!\mathfrak{j}^*A = \xi_D^*\mathfrak{h}'_!\mathfrak{j}'^*A = \tilde h_!\xi'^*\mathfrak{j}'^*A = \tilde h_!\tilde j^*\xi_D^*A.$$

Taking here $A = \dot{\underline{\mathcal{L}}}_w^\sharp$ (with $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}, \mathcal{L} \in \lambda$ with $w\underline{D}\lambda = \lambda$) and using 41.1(c) we obtain $\xi_D^*\Phi_u(\dot{\underline{\mathcal{L}}}_w^\sharp) = \tilde h_!\tilde j^*(\underline{(\mathrm{Ad}(\delta^{-1})^*\mathcal{L})}_w^\sharp)$ or equivalently $\xi_D^*\Phi_u(\dot{\underline{\mathcal{L}}}_w^\sharp) = \bar\sigma_!\tilde j'^*((\mathrm{Ad}(\delta^{-1})^*\mathcal{L})_w^\sharp)$ where
$\bar X = \{(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \in \tilde C; h_2^{-1}h_3 \in \overline{B^*\dot wB^*}\}$
and $\tilde j' : \bar X \to \bar C_w, \bar\sigma : \bar X \to C$ are the restrictions of $\tilde j, \tilde h$. Let
$\bar X_0 = \{(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \in \tilde C; h_2^{-1}h_3 \in B^*\dot wB^*\}$
and let $\tilde j'_0 : \bar X_0 \to C_w$ be the restriction of $\tilde j$. Let $\mathcal{F}_0 = \tilde j_0'^*(\mathrm{Ad}(\delta^{-1})^*\mathcal{L})$, a local system on $\bar X_0$. Since $\tilde j'$ is a fibration with smooth connected fibres we have $\tilde j'^*((\mathrm{Ad}(\delta^{-1})^*\mathcal{L})_w^\sharp) = IC(\bar X, \mathcal{F}_0)$. Thus, $\xi_D^*\Phi_u(\dot{\underline{\mathcal{L}}}_w^\sharp) = \bar\sigma_!(IC(\bar X, \mathcal{F}_0))$. From the definitions we see that $\mathcal{F}_0 = \bar\tau^*\mathcal{L}''$ hence $\bar\sigma_!(IC(\bar X, \mathcal{F}_0)) = \bar K$ and

(a) $$\xi_D^*\Phi_u(\dot{\underline{\mathcal{L}}}_w^\sharp) = \bar K$$

where $\bar\tau^*\mathcal{L}', \bar K$ are given as in 40.10 in terms of
$(u^{-1}, \mathcal{L}), (w, \mathrm{Ad}(\delta^{-1})^*\mathcal{L}), (\epsilon(u), \mathrm{Ad}(\delta\dot u\delta^{-1})^*\mathrm{Ad}(\delta^{-1})^*\mathcal{L})$
instead of $(w, \mathcal{L}), (w', \mathcal{L}'), (w'', \mathcal{L}'')$.

41.3. For $J \subset \mathbf{I}$ let $\mathcal{D}^{cs}_J(C)$ be the subcategory of $\mathcal{D}^{cs}(C)$ whose objects are those $K \in \mathcal{D}(C)$ such that for any j, any simple subquotient of ${}^pH^jK$ is isomorphic to $\underline{\mathcal{L}}^\sharp_w$ for some $\mathcal{L} \in \mathfrak{s}$ and some $w \in \mathbf{W}_J$.

Let $J, J' \subset \mathbf{I}$. Let $K \in \mathcal{D}^{cs}_J(C), K' \in \mathcal{D}^{cs}_{J'}(C)$, and let $w', w'' \in \mathbf{W}$, $\lambda', \lambda'' \in \underline{\mathfrak{s}}$, $\mathcal{L}' \in \lambda', \mathcal{L}'' \in \lambda''$. Let $A = \underline{\mathcal{L}''_{w''}}^\sharp[d_{w''}]$. We show:

(a) *If (i) $A \dashv K * \underline{\mathcal{L}'_{w'}}^\sharp[d_{w'}]$ or (ii) $A \dashv \underline{\mathcal{L}'_{w'}}^\sharp[d_{w'}] * K'$ or (iii) $A \dashv K * \underline{\mathcal{L}'_{w'}}^\sharp[d_{w'}] * K'$ then $(w'', \lambda'') \preceq_{J,J'} (w', \lambda')$.*

For the proof we may assume that $\mathbf{k}$ is an algebraic closure of a finite field. Then the results in 40.7 are applicable. We first consider the case (i). In this case we can find $\mathcal{L} \in \mathfrak{s}, w \in \mathbf{W}_J$ such that $A \dashv \underline{\mathcal{L}}^\sharp_w[d_w] * \underline{\mathcal{L}'_{w'}}^\sharp[d_{w'}]$. By 40.11(a), $[w'', \lambda'']$ appears with non-zero coefficient in the expansion of the product $[w, \lambda] * [w', \lambda']$ in terms of the basis $([y, \nu])$ of $\mathfrak{K}(C)$. Applying ω (see 40.7(b)) we see that $c_{w'',\lambda''}$ appears with non-zero coefficient in the expansion of the product $c_{w,\lambda}c_{w',\lambda'}$ in terms of the basis $(c_{y,\nu})$ of H and the desired result follows. Case (ii) is treated in an entirely similar way. We now consider case (iii). In this case we must have $A \dashv A' * K'$ for some simple perverse sheaf A' such that $A' \dashv K * \underline{\mathcal{L}'_{w'}}^\sharp[d_{w'}]$. We have $A' = \underline{\mathcal{M}}^\sharp_y[d_y]$ where $y \in \mathbf{W}$, $\mathcal{M} \in \mathfrak{s}$. Let ν be the isomorphism class of $\mathcal{M}$. From case (ii) applied to $A \dashv A' * K'$ we see that $(w'', \lambda'') \preceq_{J,J'} (y, \nu)$. From case (i) applied to $A' \dashv K * \underline{\mathcal{L}'_{w'}}^\sharp[d_{w'}]$ we see that $(y, \nu) \preceq_{J,J'} (w', \lambda')$. Combining these two inequalities we obtain $(w'', \lambda'') \preceq_{J,J'} (w', \lambda')$, as desired.

41.4. Let $J \subset \mathbf{I}$. In the remainder of this section we write $\mathfrak{f}, \mathfrak{e}$ instead of $\mathfrak{f}_{\emptyset,J} : \mathcal{D}(Z_{\emptyset,D} \to \mathcal{D}(Z_{J,D})$, $\mathfrak{e}_{\emptyset,J} : \mathcal{D}(Z_{J,D} \to \mathcal{D}(Z_{\emptyset,D})$. We note:

(a) *If $A \in \mathcal{D}(Z_{J,D})$ then $\mathfrak{f}\mathfrak{e}(A) \cong A[m] \oplus A'$ for some $m \in \mathbf{Z}$ and some $A' \in \mathcal{D}(Z_{J,D})$.*

See [Gi], [MV] for the special case $D = G^0, J = \mathbf{I}$ and [L10, 6.6] for the general case. We show:

(b) *Let A be a simple perverse sheaf on $Z_{J,D}$. Then $A \dashv \mathfrak{f}({}^pH^j(\mathfrak{e}(A)))$ for some $j \in \mathbf{Z}$.*

Assume that this is not true. As in [BBD, p.142], for any $n \in \mathbf{Z}$ we have a distinguished triangle $({}^p\tau_{\le n-1}\mathfrak{e}A, {}^p\tau_{\le n}\mathfrak{e}A, {}^pH^n(\mathfrak{e}A)[-n])$ hence a distinguished triangle

$(\mathfrak{f}({}^p\tau_{\le n-1}\mathfrak{e}A), \mathfrak{f}({}^p\tau_{\le n}\mathfrak{e}A), \mathfrak{f}({}^pH^n(\mathfrak{e}A))[-n])$.

Using our assumption, we see that $A \dashv \mathfrak{f}({}^p\tau_{\le n-1}\mathfrak{e}A)$ if and only if $A \dashv \mathfrak{f}({}^p\tau_{\le n}\mathfrak{e}A)$. Thus we have $A \dashv \mathfrak{f}({}^p\tau_{\le n}\mathfrak{e}A)$ for some n if and only if $A \dashv \mathfrak{f}({}^p\tau_{\le n}\mathfrak{e}A)$ for any n. Since ${}^p\tau_{\le n}\mathfrak{e}A = 0$ for some n we see that $A \not\dashv \mathfrak{f}({}^p\tau_{\le n}\mathfrak{e}A)$ for any n. Since ${}^p\tau_{\le n}\mathfrak{e}A = \mathfrak{e}A$ for some n we deduce that $A \not\dashv \mathfrak{f}\mathfrak{e}A$. This contradicts (a); (b) is proved.

We show:

(c) *If A is a simple perverse sheaf on $Z_{J,D}$ then there exists a simple perverse sheaf A' on $Z_{\emptyset,D}$ such that $A \dashv \mathfrak{f}(A'), A' \dashv \mathfrak{e}(A)$.*

By (b) we can find $i, j \in \mathbf{Z}$ such that $A \dashv {}^pH^i(\mathfrak{f}(P))$ where $P = {}^pH^j(\mathfrak{e}(A))$.

Assume that $A \not\dashv {}^pH^i(\mathfrak{f}(A'))$ for any simple subquotient A' of P. We claim that $A \not\dashv {}^pH^i(\mathfrak{f}(P'))$ for any subobject P' of P. We argue by induction on

the length of P'. If P' has length 1 the claim holds by assumption. If P' has length ≥ 2, we can find a simple subobject P'' of P'. We have a distinguished triangle $(\mathfrak{f}(P''), \mathfrak{f}(P'), \mathfrak{f}(P'/P''))$. Hence we have an exact sequence ${}^pH^i(\mathfrak{f}(P'')) \to {}^pH^i(\mathfrak{f}(P')) \to {}^pH^i(\mathfrak{f}(P'/P''))$. By the induction hypothesis, we have $A \not\dashv {}^pH^i(\mathfrak{f}(P'')), A \not\dashv {}^pH^i(\mathfrak{f}(P'/P''))$. Hence $A \not\dashv {}^pH^i(\mathfrak{f}(P'))$. This proves the claim. In particular, $A \not\dashv {}^pH^i(\mathfrak{f}(P))$, contradicting the definition of i, P.

We see that there exists a simple subquotient A' of P such that $A \dashv {}^pH^i(\mathfrak{f}(A'))$. Then A' is as required by (c).

Let $\bar{d}_w = \dim Z^w_{\emptyset,D}$. Let

(d) $A' = \underline{\dot{\mathcal{L}}}^\sharp_w[\bar{d}_w], A'' = \underline{\dot{\mathcal{M}}}^\sharp_y[\bar{d}_y] \in \hat{Z}_{\emptyset,D}, \mathcal{L} \in \lambda, \mathcal{M} \in \nu$.

Here $w\underline{D}\lambda = \lambda, y\underline{D}\nu = \nu$. We show:

(e) *Let A be a character sheaf on $Z_{J,D}$ such that $A \dashv \mathfrak{f}(A'), A'' \dashv \mathfrak{e}(A)$. Then $(y, \underline{D}\nu) \preceq_{J,J'} (w, \underline{D}\lambda)$.*

Since $\mathfrak{f}$ is proper, $\mathfrak{f}(A')$ is a semisimple complex (see [BBD]). Hence $\mathfrak{f}(A') \cong A[m] \oplus A_1$ for some $m \in \mathbf{Z}, A' \in \mathcal{D}(Z_{J,D})$ and $\mathfrak{e}\mathfrak{f}(A') \cong \mathfrak{e}(A)[m] \oplus \mathfrak{e}(A_1)$. Hence from $A'' \dashv \mathfrak{e}(A)$ we can deduce $A'' \dashv \mathfrak{e}\mathfrak{f}(A')$. By 37.2 we have $\mathfrak{e}\mathfrak{f}(A') \approx \{\Phi_u(A')[[-m_u]]; u \in \mathbf{W}_J\}$ where m_u are certain integers. Hence for some $u \in \mathbf{W}_J$ we have $A'' \dashv \Phi_u(A')[[-m_u]]$ that is, $A'' \dashv \Phi_u(A')$ and $\xi_D^* A''[\mathbf{r}] \dashv \xi_D^* \Phi_u(A')[\mathbf{r}]$. Hence using 41.2(a) we have $\xi_D^* A''[\mathbf{r}] \dashv \bar{K}$ where $\bar{K}$ is as in the end of 41.2. Thus, $\underline{\mathcal{M}}^\sharp_y[d_y] \dashv \bar{K}$. Using 40.10(b) we deduce that

$$\underline{\mathcal{M}}^\sharp_y[d_y] \dashv \underline{\mathrm{Ad}(\dot{w})^{-1})^*\mathrm{Ad}(\delta^{-1})^*\mathcal{L}}_{u^{-1}} * \underline{(\mathrm{Ad}(\delta^{-1})^*\mathcal{L})}^\sharp_w * \underline{\mathrm{Ad}(\delta\dot{u}\delta^{-1})^*\mathrm{Ad}(\delta^{-1})^*\mathcal{L}}_{\epsilon(u)}.$$

Using this and 41.3(a) we see that (e) holds.

We show:

(f) *Let A be a character sheaf on $Z_{J,D}$. In the setup of (d) assume that $A \dashv \mathfrak{f}(A')$, $A' \dashv \mathfrak{e}(A)$, $A \dashv \mathfrak{f}(A'')$, $A'' \dashv \mathfrak{e}(A)$. Then $(y, \underline{D}\nu) \sim_{J,J'} (w, \underline{D}\lambda)$.*

Applying (e) to A', A'' we see that $(y, \underline{D}\nu) \preceq_{J,J'} (w, \underline{D}\lambda)$. Applying (e) to A'', A' (instead of A', A'') we see that $(w, \underline{D}\lambda) \preceq_{J,J'} (y, \underline{D}\nu)$. Hence (f) holds.

From (c),(f) we see that there is a well defined map $A \mapsto \mathbf{c}_A$ from the set of character sheaves on $Z_{J,D}$ (up to isomorphism) to the set of (J, J')-two-sided cells in $\mathbf{W} \times \underline{\mathcal{F}}$ where $\mathbf{c}_A$ is the unique (J, J')-two-sided cell that contains

$$\{(w, \underline{D}\lambda) \in \mathbf{W} \times \underline{\mathfrak{s}}; w\underline{D}\lambda = \lambda, A \dashv \mathfrak{f}(\underline{\dot{\mathcal{L}}}^\sharp_w[\bar{d}_w]), \underline{\dot{\mathcal{L}}}^\sharp_w[\bar{d}_w] \dashv A\}$$

(a non-empty set); here $\mathcal{L} \in \lambda$.

41.5. In the setup of 41.4, let A be a character sheaf on $Z_{J,D}$. We show:

(a) *There exists $(w, \underline{D}\lambda) \in \mathbf{c}_A$ such that $w\underline{D}\lambda = \lambda$, $A \dashv \mathfrak{f}(\underline{\dot{\mathcal{L}}}^\sharp_w[\bar{d}_w])$. If $(w', \underline{D}\lambda') \in \mathbf{W} \times \underline{\mathfrak{s}}$ is such that $w'\underline{D}\lambda' = \lambda'$, $A \dashv \mathfrak{f}(\underline{\dot{\mathcal{L}}'}^\sharp_{w'}[\bar{d}_{w'}])$ then $(w, \underline{D}\lambda) \preceq_{J,J'} (w', \underline{D}\lambda')$. Here $\mathcal{L} \in \lambda, \mathcal{L}' \in \lambda'$.*

(b) *There exists* $(w,\underline{D}\lambda)\in\mathbf{c}_A$ *such that* $w\underline{D}\lambda=\lambda$, $\dot{\underline{\mathcal{L}}}_w^\sharp[\bar{d}_w]\dashv\mathfrak{e}(A)$. *If* $(w',\underline{D}\lambda')\in\mathbf{W}\times\underline{\mathfrak{s}}$ *is such that* $w'\underline{D}\lambda'=\lambda'$, $\dot{\underline{\mathcal{L}}}'_{w'}{}^\sharp[\bar{d}_{w'}]\dashv\mathfrak{e}(A)$ *then* $(w',\underline{D}\lambda')\preceq_{J,J'}(w,\underline{D}\lambda)$. *Here* $\mathcal{L}\in\lambda,\mathcal{L}'\in\lambda'$.
Note that (a) follows immediately from 41.4(c),(e) and the definition of $\mathbf{c}_A$. Similarly, (b) follows from 41.4(c),(e) and the definition of $\mathbf{c}_A$.

41.6. In this subsection we assume that $J=\mathbf{I}$. The $\mathcal{A}$ linear map $H\longrightarrow H$ given by

(a) $\tilde{T}_w 1_\lambda\mapsto\tilde{T}_{\epsilon(w)}1_{\underline{D}\lambda}$ for $w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}$

is an $\mathcal{A}$-algebra isomorphism. It carries $c_{w,\lambda}$ to $c_{\epsilon(w),\underline{D}\lambda}$ for any $w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}$. It induces a bijection $\mathbf{c}\mapsto\mathbf{c}'$ from the set of two-sided cells in $\mathbf{W}\times\underline{\mathfrak{s}}$ onto itself. We show:

(b) *If* A *is a character sheaf on* D *then* $(\mathbf{c}_A)'=\mathbf{c}_A$.

Consider the automorphism $\mathrm{Ad}(\delta):D\longrightarrow D$. From the definitions we see that for $(w,\lambda)\in\mathbf{W}\times\underline{\mathfrak{s}}$ such that $w\underline{D}\lambda=\lambda$ we have $A\dashv\mathfrak{f}(\dot{\underline{\mathcal{L}}}_w^\sharp[\bar{d}_w])$ if and only if $\mathrm{Ad}(\delta^{-1})^*A\dashv\mathfrak{f}(\underline{\mathrm{Ad}(\dot{\underline{D}}^{-1})^*\mathcal{L}}^\sharp_{\epsilon(w)}[\bar{d}_w])$. Using this and 41.5(a) we see that

$\mathbf{c}_{\mathrm{Ad}(\delta^{-1})^*A}=(\mathbf{c}_A)'$.

It is then enough to show that $\mathrm{Ad}(\delta^{-1})^*A\cong A$. By the G^0-equivariance of A we have $m^*A\cong q^*A$ where $m:G^0\times D\longrightarrow D$ is $(x,g)\mapsto xgx^{-1}$ and $q:G^0\times D\longrightarrow D$ is $(x,g)\mapsto g$. Define $r:D\longrightarrow G^0\times D$ by $r(g)=(\delta g^{-1},g)$. Then $r^*m^*A\cong r^*q^*A$ that is, $(mr)^*A\cong(qr)^*A$. We have $mr=\mathrm{Ad}(\delta)$, $qr=1$ hence $\mathrm{Ad}(\delta)^*A\cong A$ and $\mathrm{Ad}(\delta^{-1})^*A\cong A$, as required.

Note also that for (w,λ) as above we have:

(c) $\mathfrak{f}(\underline{\mathrm{Ad}(\dot{\underline{D}}^{-1})^*\mathcal{L}}^\sharp_{\epsilon(w)}[\bar{d}_w])\cong\mathfrak{f}(\dot{\underline{\mathcal{L}}}_w^\sharp[\bar{d}_w])$.

Indeed, let $K=\mathfrak{f}(\dot{\underline{\mathcal{L}}}_w^\sharp[\bar{d}_w])$. Clearly we have $m^*K\cong q^*K$ with m,q as above. Then as in the proof of (b) we see that $\mathrm{Ad}(\delta)^*K\cong K$. From the definitions we see that $\mathfrak{f}(\underline{\mathrm{Ad}(\dot{\underline{D}}^{-1})^*\mathcal{L}}^\sharp_{\epsilon(w)}[\bar{d}_w])=\mathrm{Ad}(\delta^{-1})^*K$. Since $\mathrm{Ad}(\delta^{-1})^*K\cong K$, (c) follows.

41.7. In this and next subsection we assume that $\mathbf{k}$ is an algebraic closure of a finite field. From 41.1(c) we see that $\xi_D^*:\mathcal{D}(Z_{\emptyset,D})\longrightarrow\mathcal{D}(C)$ restricts to a functor $\mathcal{D}^{cs}(Z_{\emptyset,D})\longrightarrow\mathcal{D}^{cs}(C)$ hence, as in 36.8, the $\mathcal{A}$-linear map $gr(\xi_D^*):\mathfrak{K}(Z_{\emptyset,D})\longrightarrow\mathfrak{K}(C)$ is well defined; from 41.1(c) we see also that

(a) $gr(\xi_D^*)(\dot{\underline{\mathcal{L}}}_w^\sharp[\bar{d}_w])=(-v)^{\mathbf{r}}[w;\underline{D}\lambda]$

for $w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}$ such that $w\underline{D}\lambda=\lambda$ and $\mathcal{L}\in\lambda$. From (a) we see that $gr(\xi_D^*)$ is injective with image equal to $\mathfrak{K}(C)^D$, the $\mathcal{A}$-submodule of $\mathfrak{K}(C)$ spanned by $\{[w;\underline{D}\lambda];w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}},w\underline{D}\lambda=\lambda\}$ or equivalently by $\{[w;\underline{D}\lambda]';w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}},w\underline{D}\lambda=\lambda\}$. Thus, $gr(\xi_D^*)$ defines an isomorphism $\eta':\mathfrak{K}(Z_{\emptyset,D})\xrightarrow{\sim}\mathfrak{K}(C)^D$. Let $\eta=\eta'^{-1}$.

Let $n\in\mathbf{N}_{\mathbf{k}}^*$. Let $\mathfrak{K}(C)_n^D$ be the $\mathcal{A}$-submodule of $\mathfrak{K}(C)$ spanned by $\{[w;\underline{D}\lambda];w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n,w\underline{D}\lambda=\lambda\}$ or equivalently by $\{[w;\underline{D}\lambda]';w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n,w\underline{D}\lambda=\lambda\}$.

Let $u, w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n$ be such that $w\underline{D}\lambda = \lambda$ and let $\mathcal{L} \in \lambda$. From 37.3(c) we see that the $\mathcal{A}$-linear map $gr(\Phi_u) : \mathfrak{K}(Z_{\emptyset,D}) \longrightarrow \mathfrak{K}(Z_{\emptyset,D})$ is well defined; we denote it again by Φ_u. From 40.10(a), 41.2(a) we have

$$[u^{-1}; \lambda]' * [w; \underline{D}\lambda]'^\sharp * [\epsilon_D(u); \underline{D}(u^{-1}\lambda)]' = (v^2 - 1)^{2\mathbf{r}} \eta' \Phi_u \eta([w; \underline{D}\lambda]'^\sharp),$$

equality in $\mathfrak{K}(C)$. If $\lambda' \in \underline{\mathfrak{s}}_n, \lambda' \neq \lambda$ we have (from 40.7) that $[u^{-1}; \lambda']' * [w; \underline{D}\lambda]'^\sharp * [\epsilon_D(u); \underline{D}(u^{-1}\lambda')]' = 0$. It follows that

$$(v^2 - 1)^{2\mathbf{r}} \eta' \Phi_u \eta([w, \underline{D}\lambda]'^\sharp)) = \sum_{\lambda' \in \underline{\mathfrak{s}}_n} [u^{-1}; \lambda']' * [w; \underline{D}\lambda]'^\sharp * [\epsilon_D(u); \underline{D}(u^{-1}\lambda')]'$$

Using this and the definition of $\mathfrak{K}(C)_n^D$ we see that

$$(v^2 - 1)^{2\mathbf{r}} \eta' \Phi_u \eta(x) = \sum_{\lambda' \in \underline{\mathfrak{s}}_n} [u^{-1}; \lambda']' * x * [\epsilon_D(u); \underline{D}(u^{-1}\lambda')]'$$

for any $x \in \mathfrak{K}(C)_n^D$. Applying η to both sides we obtain

(b) $$(v^2 - 1)^{2\mathbf{r}} \Phi_u \eta'(x) = \sum_{\lambda' \in \underline{\mathfrak{s}}_n} \eta([u^{-1}; \lambda']' * x * [\epsilon_D(u); \underline{D}(u^{-1}\lambda')]')$$

for any $x \in \mathfrak{K}(C)_n^D$.

41.8. In the setup of 41.4, let A be a character sheaf on $Z_{J,D}$. From 36.9(b) we see that the condition that, if $(w', \underline{D}\lambda') \in \mathbf{W} \times \underline{\mathfrak{s}}$ is such that $w'\underline{D}\lambda' = \lambda'$, then we have $A \dashv \mathfrak{f}(\dot{\underline{\mathcal{L}}}'_{w'}{}^\sharp[\bar{d}_{w'}])$ if and only if A appears with coefficient $\neq 0$ in the expansion of $\mathfrak{f}(\dot{\underline{\mathcal{L}}}'_{w'}{}^\sharp[\bar{d}_{w'}]) \in \mathfrak{K}(Z_{J,D})$ as a linear combination of the canonical basis of $\mathfrak{K}(Z_{J,D})$. Hence from 41.5(a) we deduce:

(a) *There exists $(w, \underline{D}\lambda) \in \mathbf{c}_A$ such that $w\underline{D}\lambda = \lambda$ and A appears with non-zero coefficient in $\mathfrak{f}(\dot{\underline{\mathcal{L}}}_w^\sharp[\bar{d}_w]) \in \mathfrak{K}(Z_{J,D})$. If $(w', \underline{D}\lambda') \in \mathbf{W} \times \underline{\mathfrak{s}}$ is such that $w'\underline{D}\lambda' = \lambda'$ and A appears with non-zero coefficient in $\mathfrak{f}(\dot{\underline{\mathcal{L}}}'_{w'}{}^\sharp[\bar{d}_{w'}]) \in \mathfrak{K}(Z_{J,D})$ then $(w, \underline{D}\lambda) \preceq_{J,J'} (w', \underline{D}\lambda')$. Here $\mathcal{L} \in \lambda, \mathcal{L}' \in \lambda'$.*

Clearly, property (a) characterizes $\mathbf{c}_A$.

41.9. Let $J \subset J' \subset \mathbf{I}$ and let D' be another connected component of G. Let $A_0 \in \mathcal{D}(Z_{J,D})$, $A' \in \mathcal{D}(Z_{\epsilon_D(J'),D'})$. We show:

(a) $\mathfrak{f}_{J,J'}(A_0) * A' \cong \mathfrak{f}_{J,J'}(A_0 * \mathfrak{e}_{\epsilon_D(J),\epsilon_D(J')} A')$ in $\mathcal{D}(Z_{J',D'D})$.

Indeed, from the definitions we see that both sides of (a) can be identified with $b_! c^*(A_0 \boxtimes A')$ where b, c are as in the diagram

$$Z_{J,D} \times Z_{\epsilon_D(J'),D'} \xleftarrow{c} Y \xrightarrow{b} Z_{J',D'D}$$

where

$$\begin{aligned} Y = &\{(P, R, R', gU_R, g'U_{R'}); P \in \mathcal{P}_J, R \in \mathcal{P}_{J'}, R' \in \mathcal{P}_{\epsilon_D(J')}, \\ &gU_R \in D/U_R, g'U_{R'} \in D'/U_{R'}, gRg^{-1} = R', P \subset R\}, \end{aligned}$$

c is $(P, R, R', gU_R, g'U_{R'}) \mapsto ((P, gU_P), (R', g'U_{R'}),$

b is $(P, R, R', gU_R, g'U_{R'}) \mapsto (R, g'gU_R)$.

An entirely similar proof shows that, if $A \in \mathcal{D}(Z_{J',D})$, $A'_0 \in \mathcal{D}(Z_{\epsilon_D(J),D'})$ then

(b) $A * \mathfrak{f}_{\epsilon_D(J),\epsilon_D(J')}(A'_0) \cong \mathfrak{f}_{J,J'}(\mathfrak{e}_{J,J'} A * A'_0)$ in $\mathcal{D}(Z_{J',D'D})$.

41.10. Let $\mathbf{c}$ be a two-sided cell in $\mathbf{W} \times \underline{\mathfrak{s}}$. Let $\bar{\mathbf{c}}$ be the set of all $(w,\lambda) \in \mathbf{W} \times \underline{\mathfrak{s}}$ such that $(w,\lambda) \preceq_{\mathbf{I},\mathbf{I}} (y,\nu)$ for some/any $(y,\nu) \in \mathbf{c}$.

If $K \in \mathcal{D}(Z_{\emptyset,D})$, we say that $K \in \mathcal{D}^{cs}_{\bar{\mathbf{c}}}(Z_{\emptyset,D})$ if for any $j \in \mathbf{Z}$ and simple subquotient A of ${}^pH^j(K)$ satisfies $\mathbf{c}_A \subset \bar{\mathbf{c}}$.

Let D' be another connected component of G. We show:

(a) *If $K \in \mathcal{D}^{cs}_{\bar{\mathbf{c}}}(Z_{\emptyset,D})$, $K' \in \mathcal{D}^{cs}(Z_{\epsilon_D(J'),D'})$, then $K * K' \in \mathcal{D}^{cs}_{\bar{\mathbf{c}}}(Z_{\emptyset,D'D})$.*

We may assume that $\mathbf{k}$ is an algebraic closure of a finite field. We may assume that $K \in \hat{Z}_{\emptyset,D}$ and $\mathbf{c}_K \subset \bar{\mathbf{c}}$. Then there exists $(w,\underline{D}\lambda) \in \mathbf{c}_K$ such that $w\underline{D}\lambda = \lambda$, $K \dashv \mathfrak{f}(\underline{\dot{\mathcal{L}}}^{\sharp}_w[\bar{d}_w])$, $\mathcal{L} \in \lambda$. It is enough to show that, if $\tilde{A} \in \hat{Z}_{\emptyset,D'D}$ is such that $\tilde{A} \dashv K * K'$ then $\mathbf{c}_{\tilde{A}} \subset \bar{\mathbf{c}}$. Since $\mathfrak{f}(\underline{\dot{\mathcal{L}}}^{\sharp}_w[\bar{d}_w])$ is a semisimple complex (see the line after 41.4(e)) we have $\mathfrak{f}(\underline{\dot{\mathcal{L}}}^{\sharp}_w[\bar{d}_w]) \cong K[m] \oplus \tilde{K}$ for some $m \in \mathbf{Z}$, $\tilde{K} \in \mathcal{D}(Z_{\emptyset,D'D})$. It follows that $\mathfrak{f}(\underline{\dot{\mathcal{L}}}^{\sharp}_w[\bar{d}_w]) * K' \cong K * K'[m] \oplus \tilde{K} * K'$ hence $\tilde{A} \dashv \mathfrak{f}(\underline{\dot{\mathcal{L}}}^{\sharp}_w[\bar{d}_w]) * K'$. By 41.9(a) we have $\mathfrak{f}(\underline{\dot{\mathcal{L}}}^{\sharp}_w[\bar{d}_w]) * K' \cong \mathfrak{f}(\underline{\dot{\mathcal{L}}}^{\sharp}_w[\bar{d}_w] * \mathfrak{e}(K'))$ hence $\tilde{A} \dashv \mathfrak{f}(\underline{\dot{\mathcal{L}}}^{\sharp}_w[\bar{d}_w] * \mathfrak{e}(K'))$. We deduce that there exists $K'_0 \in \hat{Z}_{\emptyset,D'}$ such that $\tilde{A} \dashv \mathfrak{f}(\underline{\dot{\mathcal{L}}}^{\sharp}_w[\bar{d}_w] * K'_0)$ and $K''_0 \in \hat{Z}_{\emptyset,D'D}$ such that $K''_0 \dashv \underline{\dot{\mathcal{L}}}^{\sharp}_w[\bar{d}_w] * K'_0$, $\tilde{A} \dashv \mathfrak{f}(K''_0)$. We then have $\xi^*_{D'D}K''_0[\mathbf{r}] \dashv \xi^*_{D'D}(\underline{\dot{\mathcal{L}}}^{\sharp}_w[\bar{d}_w] * K'_0)[\mathbf{r}]$, hence, using 41.1(d), $\xi^*_{D'D}K''_0[\mathbf{r}] \dashv (\xi^*_D\underline{\dot{\mathcal{L}}}^{\sharp}_w[\bar{d}_w]) * \xi_{D'}K'_0)$. Setting $gr(\xi^*_{D'D}K''_0[\mathbf{r}]) = [w_1, \underline{D}'\underline{D}\lambda_1] \in \mathfrak{K}(C)$ with $(w_1,\lambda_1) \in \mathbf{W}\underline{\mathfrak{s}}$ we see, using 41.3(a) that $(w_1,\underline{D}'\underline{D}\lambda_1) \preceq_{\mathbf{I},\mathbf{I}} (w,\underline{D}\lambda)$. From $\tilde{A} \dashv \mathfrak{f}(K''_0)$ we see using 41.5(a) that $\mathbf{c}_{\tilde{A}} \preceq_{\mathbf{I},\mathbf{I}} (w_1,\underline{D}'\underline{D}\lambda_1)$ (that is, some/any element of $\mathbf{c}_{\tilde{A}}$ is $\preceq_{\mathbf{I},\mathbf{I}} (w_1,\underline{D}'\underline{D}\lambda_1)$). Using the transitivity of $\preceq_{\mathbf{I},\mathbf{I}}$ we see that $\mathbf{c}_{\tilde{A}} \preceq_{\mathbf{I},\mathbf{I}} (w,\underline{D}\lambda)$. This proves (a).

An entirely similar argument shows:

(b) *If $K \in \mathcal{D}^{cs}(Z_{\emptyset,D})$, $K' \in \mathcal{D}^{cs}_{\bar{\mathbf{c}}}(Z_{\epsilon_D(J'),D'})$, then $K * K' \in \mathcal{D}^{cs}_{\bar{\mathbf{c}}}(Z_{\emptyset,D'D})$.*

42. Duality and the functor $\mathfrak{f}_{\emptyset,\mathbf{I}}$

42.1. In this section we fix a connected component D of G. We write ϵ instead of $\epsilon_D : \mathbf{W} \to \mathbf{W}$. We write $\mathfrak{f}$ instead of $\mathfrak{f}_{\emptyset,J} : \mathcal{D}(Z_{\emptyset,D} \to \mathcal{D}(Z_{J,D})$. We assume that $\mathbf{k}$ is an algebraic closure of a finite field.

Let $J \subset \mathbf{I}$ be such that $\epsilon(J) = J$. Recall from 30.3 that $V_{J,D} = \{(P, gU_P); P \in \mathcal{P}_J, gU_P \in N_DP/U_P\}$. As in 30.4 (with $J' = \mathbf{I}$) we consider the diagram $V_{J,D} \xleftarrow{c} V_{J,\mathbf{I},D} \xrightarrow{d} D$ where $V_{J,\mathbf{I},D} = \{(P,g); P \in \mathcal{P}_J, g \in N_DP\}$, c is $(P,g) \mapsto (P, gU_P)$ and d is $(P,g) \mapsto g$. Define $\tilde{f}_J : \mathcal{D}(V_{J,D}) \to \mathcal{D}(D)$, $\tilde{e}_J : \mathcal{D}(D) \to \mathcal{D}(V_{J,D})$ by $\tilde{f}_JA = d_!c^*A, \tilde{e}_JA' = c_!d^*A'$. (In the notation of 30.4 we have $\tilde{f}_J = \tilde{f}_{J,\mathbf{I}}$, $\tilde{e}_J = \tilde{e}_{J,\mathbf{I}}$.) Define $f_J : \mathcal{D}(V_{J,D}) \to \mathcal{D}(D)$, $e_J : \mathcal{D}(D) \to \mathcal{D}(V_{J,D})$ by $f_JA = \tilde{f}_JA[[\alpha_J/2]]$, $e_JA = \tilde{e}_JA[[\alpha_J/2]]$ where $\alpha_J = \dim \mathcal{P}_J$. (In the notation of 30.4 we have $f_JA = f_{J,\mathbf{I}}A(\alpha_J/2)$, $e_JA = e_{J,\mathbf{I}}A(-\alpha_J/2)$. Thus, f_J, e_J are the same, up to a twist, as $f_{J,\mathbf{I}}, e_{J,\mathbf{I}}$.)

From 30.5 (with $J' = \mathbf{I}$) we see that for $A \in \mathcal{D}(V_{J,D})$, $A' \in \mathcal{D}(D)$ we have canonically

(a) $\mathrm{Hom}_{\mathcal{D}(V_{J,D})}(e_JA', A) = \mathrm{Hom}_{\mathcal{D}(D)}(A', f_JA)$.

Let $CS(V_{J,D}), CS(D)$ be as in 38.1. From 38.2, 38.3 we see that

(b) *f_J, e_J restrict to functors $CS(V_{J,D}) \to CS(D)$, $CS(D) \to CS(V_{J,D})$ denoted again by f_J, e_J.*
We show:

(c) *if $A \in CS(V_{J,D})$ comes from a pure complex of weight 0 with respect to a rational structure over a finite subfield of $\mathbf{k}$ then $f_J A$ (naturally regarded as a mixed complex) is pure of weight 0.*
Indeed, the functor c^* preserves pure complexes of weight 0 since c is smooth with connected fibres; the functor $d_!$ preserves pure complexes of weight 0 since d is proper (see [D, 6.2.6]) and $[[\alpha_J]]$ also preserves pure complexes of weight 0.

We show:

(d) *if $A' \in CS(D)$ comes from a pure complex of weight 0 with respect to a rational structure over a finite subfield of $\mathbf{k}$ then $e_J A'$ (naturally regarded as a mixed complex) is pure of weight 0.*
Using (b), it is enough to show that for any simple A as in (c), the natural action of Frobenius on the vector space $\mathrm{Hom}_{\mathcal{D}(V_{J,D})}(e_J A', A)$ has weight 0. Using (a) we see that it is enough to show that the natural action of Frobenius on the vector space $\mathrm{Hom}_{\mathcal{D}(D)}(A', f_J A)$ has weight 0. This follows from (c) using (b).

Define an imbedding $s : V_{J,D} \to Z_{J,D}$ by $(P, gU_P) \mapsto (P, P, gU_P)$. From the definitions we see that

(e) *$\tilde{f}_J : \mathcal{D}(V_{J,D}) \to \mathcal{D}(D)$ is the composition $\mathcal{D}(V_{J,D}) \xrightarrow{s_!} \mathcal{D}(Z_{J,D}) \xrightarrow{\mathfrak{f}_{J,\mathbf{I}}} \mathcal{D}(D)$,*

(f) *$\tilde{e}_J : \mathcal{D}(D) \to \mathcal{D}(V_{J,D})$ is the composition $\mathcal{D}(D) \xrightarrow{\mathfrak{e}_{J,\mathbf{I}}} \mathcal{D}(Z_{J,D}) \xrightarrow{s^*} \mathcal{D}(V_{J,D})$.*
Let $Y = \{(B, B', gU_B) \in Z_{\emptyset,D}; \mathrm{pos}(B, B') \in \mathbf{W}_J\}$ and let $r : Y \to Z_{\emptyset,D}$ be the inclusion. From the definitions we have

(g) $s_! s^* \mathfrak{f}_{\emptyset,J} = \mathfrak{f}_{\emptyset,J} r_! r^* : \mathcal{D}(Z_{\emptyset,D}) \to \mathcal{D}(Z_{J,D})$.
Note that $V_{J,D} = {}^1Z_{J,D}$, see 36.2; hence the "character sheaves" on $V_{J,D} = {}^1Z_{J,D}$ are defined as in 36.8 and $\mathcal{D}^{cs}(V_{J,D} = \mathcal{D}^{cs}({}^1Z_{J,D})$ is defined as 36.8. In particular, $\mathfrak{K}(V_{J,D}) = \mathfrak{K}({}^1Z_{J,D})$ is defined. Let $\mathfrak{K}_0(V_{J,D}) = \oplus_A \mathbf{Z}A \subset \mathfrak{K}(V_{J,D})$ where A runs through the character sheaves on $V_{J,D}$ (up to isomorphism).

From (b) we see that $\tilde{f}_J, \tilde{e}_J$ restrict to functors $\mathcal{D}^{cs}(V_{J,D}) \to \mathcal{D}^{cs}(D)$, $\mathcal{D}^{cs}(D) \to \mathcal{D}^{cs}(V_{J,D})$ hence the $\mathcal{A}$-linear maps $gr(\tilde{f}_J) : \mathfrak{K}(V_{J,D}) \to \mathfrak{K}(D)$, $gr(\tilde{e}_J) : \mathfrak{K}(D) \to \mathfrak{K}(V_{J,D})$ are well defined; we denote them by $\tilde{f}_J, \tilde{e}_J$. Define $f_J : \mathfrak{K}(V_{J,D}) \to \mathfrak{K}(D)$ by $f_J = v^{-\alpha_J} \tilde{f}_J$ and $e_J : \mathfrak{K}(D) \to \mathfrak{K}(V_{J,D})$ by $e_J = v^{-\alpha_J} \tilde{e}_J$. We show:

(h) *$f_J : \mathfrak{K}(V_{J,D}) \to \mathfrak{K}(D)$, $e_J : \mathfrak{K}(D) \to \mathfrak{K}(V_{J,D})$ restrict to group homomorphisms $\mathfrak{K}_0(V_{J,D}) \to \mathfrak{K}_0(D)$, $\mathfrak{K}_0(D) \to \mathfrak{K}_0(V_{J,D})$ denoted again by f_J, e_J.*
It is enough to prove the following statement. If x is a canonical basis element of $\mathfrak{K}(V_{J,D})$ (resp. $\mathfrak{K}(D)$) then $f_J(x)$ (resp. $e_J(x)$) is an $\mathbf{N}$-linear combination of canonical basis elements of $\mathfrak{K}(D)$ (resp. $\mathfrak{K}(V_{J,D})$). This is immediate from (c), (d).

Now, one checks easily that $r_! r^* : \mathcal{D}(Z_{\emptyset,D}) \to \mathcal{D}(Z_{\emptyset,D})$ restricts to a functor $\mathcal{D}^{cs}(Z_{\emptyset,D}) \to \mathcal{D}^{cs}(Z_{\emptyset,D})$. (Note that, if $w \in \mathbf{W}$, $\lambda \in \underline{\mathfrak{s}}$, $\mathcal{L} \in \lambda$ and $w\underline{D}\lambda = \lambda$, then $r_! r^*(\underline{\dot{\mathcal{L}}}_w) = \underline{\dot{\mathcal{L}}}_w$ for $w \in \mathbf{W}_J$ and $r_! r^*(\underline{\dot{\mathcal{L}}}_w) = 0$ for $w \in \mathbf{W} - \mathbf{W}_J$.) It follows that the $\mathcal{A}$-linear map $gr(r_! r^*) : \mathfrak{K}(Z_{\emptyset,D}) \to \mathfrak{K}(Z_{\emptyset,D})$ (denoted by ρ_J) is well defined.

Let $\mathfrak{K}(C)^D, \eta$ be as in 41.7. Define an $\mathcal{A}$-linear map $\tilde{\rho}_J : \mathfrak{K}(C)^D \to \mathfrak{K}(C)^D$

by $[w;\underline{D}\lambda]'\mapsto[w;\underline{D}\lambda]'$ if $w\in\mathbf{W}_J,\lambda\in\underline{\mathfrak{s}},w\underline{D}\lambda=\lambda$ and $[w;\underline{D}\lambda]'\mapsto 0$ if $w\in\mathbf{W}-\mathbf{W}_J,\lambda\in\underline{\mathfrak{s}},w\underline{D}\lambda=\lambda$. From the definitions we see that

(i) $\rho_J\eta(x)=\eta\tilde\rho_J(x)$ for all $x\in\mathfrak{K}(C)^D$.

42.2. We define an $\mathcal{A}$-linear map $\mathbf{d}:\mathfrak{K}(D)\longrightarrow\mathfrak{K}(D)$ by

$$\mathbf{d}(x)=\sum_{J;J\subset\mathbf{I};\epsilon(J)=J}(-1)^{|J_\epsilon|}f_Je_J(x)$$

where f_J,e_J are as in 42.1(h) and J_ϵ is as in 38.1. We show:

(a) *Let A be a character sheaf on D. Then $\mathbf{d}(A)=\pm A'$ where A' is a character sheaf on D. Moreover $\pm$ and A' are the same as in 38.11(a).*

For any $J\subset\mathbf{I}$ such that $\epsilon(J)=J$ let $\mathcal{K}(V_{J.D})$ be as in 38.9. We shall identify $\mathfrak{K}(V_{J,D})/(v-1)\mathfrak{K}(V_{J,D})=\mathcal{K}(V_{J,D})$ as abelian groups in such a way that the image of A_1 (a character sheaf on $V_{J,D}$) in $\mathfrak{K}(V_{J,D})/(v-1)\mathfrak{K}(V_{J,D})$ is identified with the basis element A_1 of $\mathcal{K}(V_{J,D})$. From the definitions we see that the homomorphisms

$\mathfrak{K}(D)/(v-1)\mathfrak{K}(D)\longrightarrow\mathfrak{K}(V_{J,D})/(v-1)\mathfrak{K}(V_{J,D})\longrightarrow\mathfrak{K}(D)/(v-1)\mathfrak{K}(D)$

induced by e_J,f_J in 42.1(h) are then identified with the homomorphisms

$e_{J,\mathbf{I}}:\mathcal{K}(D)\longrightarrow\mathcal{K}(V_{J,D})$, $f_{J,\mathbf{I}}:\mathcal{K}(V_{J,D})\longrightarrow\mathcal{K}(D)$

in 38.2, 38.3. It follows that the endomorphism of $\mathfrak{K}(D)/(v-1)\mathfrak{K}(D)$ induced by $\mathbf{d}:\mathfrak{K}(D)\longrightarrow\mathfrak{K}(D)$ is identified with the homomorphism $\mathcal{K}(D)\longrightarrow\mathcal{K}(D)$ denoted in 38.10(a), 38.11 again by $\mathbf{d}$. Hence we have $\mathbf{d}(A)=\pm A'+(v-1)x$ (in $\mathfrak{K}(D)$) where $\pm,A'$ are as in 38.11(a) and $x\in\mathfrak{K}(D)$. From 42.1(h) we see that $\mathbf{d}(A)\in\mathfrak{K}_0(D)$. Since $\pm A'\in\mathfrak{K}_0(D)$, we see that $(v-1)x\in\mathfrak{K}_0(D)$. Since $\mathfrak{K}_0(D)\cap(v-1)\mathfrak{K}(D)=0$, we have $(v-1)x=0$ and $x=0$. This proves (a).

42.3. We have $H=H_D\oplus H'_D$ where H_D (resp. H'_D) is the $\mathcal{A}$-submodule of H_n spanned by $\{\tilde T_w1_{\underline{D}\lambda};w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}},w\underline{D}\lambda=\lambda\}$ (resp. by $\{\tilde T_w1_{\underline{D}\lambda};w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}},w\underline{D}\lambda\ne\lambda\}$). Equivalently,

$H_D=\sum_{\lambda\in\underline{\mathfrak{s}}}1_\lambda H1_{\underline{D}\lambda}\subset H$, $H'_D=\sum_{\lambda,\lambda'\in\underline{\mathfrak{s}};\lambda\ne\lambda'}1_{\lambda'}H1_{\underline{D}\lambda}\subset H$.

Recall that $\omega:\mathfrak{K}(C)\xrightarrow{\sim}H$ is defined in 40.7(b). Define an $\mathcal{A}$-linear map $\tilde\omega:H\longrightarrow\mathfrak{K}(C)^D$ by

$\tilde\omega(y)=\omega^{-1}(y)$ if $y\in H_D$,,

$\tilde\omega(y)=0$ if $y\in H'_D$.

Then $\eta\tilde\omega(y)\in\mathfrak{K}(Z_{\emptyset,D})$ is well defined for any $y\in H$. Here η is as in 41.7.

Let $n\in\mathbf{N}^*_{\mathbf{k}}$. Let $H_{n,D}=H_D\cap H_n$. Note that $H_{n,D}$ is the $\mathcal{A}$-submodule of H_n spanned by $\{\tilde T_w1_{\underline{D}\lambda};w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n,w\underline{D}\lambda=\lambda\}$.

For $J\subset\mathbf{I}$ such that $\epsilon(J)=J$ we define an $\mathcal{A}$-linear map $\rho_{J,n}:H_{n,D}\longrightarrow H_{n,D}$ by

$\tilde T_w1_{\underline{D}\lambda}\mapsto\tilde T_w1_{\underline{D}\lambda}$ if $w\in\mathbf{W}_J,\lambda\in\underline{\mathfrak{s}}_n,w\underline{D}\lambda=\lambda$,

$\tilde T_w1_{\underline{D}\lambda}\mapsto 0$ if $w\in\mathbf{W}-\mathbf{W}_J,\lambda\in\underline{\mathfrak{s}}_n,w\underline{D}\lambda=\lambda$.

We have the following result.

Lemma 42.4. *For any $y \in H_{n,D}$ we have $\mathbf{d}(\mathfrak{f}\eta\tilde{\omega}(y)) = \mathfrak{f}\eta\tilde{\omega}(\delta(y))$ where $\delta = \sum_{J\subset \mathbf{I};\epsilon(J)=J}(-1)^{|J_\epsilon|}\delta_J$ with $\delta_J : H_{n,D} \to H_{n,D}$ given by $\delta_J(y) = \rho_{J,n}(\sum_{u\in\mathbf{W}^J}\tilde{T}_{u^{-1}}y\tilde{T}_{\epsilon_D(u)})$ (the sum in the right hand side is computed in H_n but it belongs to $H_{n,D}$).*

Applying 37.2 with K, K', J repaced by $\emptyset, J, \mathbf{I}$ and with $A' \in \mathcal{D}^{cs}(Z_{\emptyset,D})$ we obtain

$\mathfrak{e}_{J,\mathbf{I}}\mathfrak{f}A' \approx \{\mathfrak{f}_{\emptyset,J}\Phi_u A'[[-m_u]]; u \in \mathbf{W}^J\}$

(in $\mathcal{D}(Z_{J,D})$, with $\Phi_u : \mathcal{D}(Z_{\emptyset,D}) \to \mathcal{D}(Z_{\emptyset,D})$ as in 37.1 and $m_u = \alpha_J - \lambda(u)$ where $\alpha_J = \dim\mathcal{P}_J$. Applying here s^* we obtain

$s^*\mathfrak{e}_{J,\mathbf{I}}\mathfrak{f}A' \approx \{s^*\mathfrak{f}_{\emptyset,J}\Phi_u A'[[-m_u]]; u \in \mathbf{W}^J\}$.

We replace $s^*\mathfrak{e}_{J,\mathbf{I}}$ by $\tilde{e}_J$ (see 42.1(f)) and we apply $\tilde{f}_J = \mathfrak{f}_{J,\mathbf{I}}s_!$ (see 42.1(e)); we obtain

$\tilde{f}_J\tilde{e}_J\mathfrak{f}A' \approx \{\mathfrak{f}_{J,\mathbf{I}}s_!s^*\mathfrak{f}_{\emptyset,J}\Phi_u A'[[-m_u]]; u \in \mathbf{W}^J\}$.

Using now 42.1(g) we obtain

$\tilde{f}_J\tilde{e}_J\mathfrak{f}A' \approx \{\mathfrak{f}_{J,\mathbf{I}}\mathfrak{f}_{\emptyset,J}r_!r^*\Phi_u A'[[-m_u]]; u \in \mathbf{W}^J\}$.

Here we replace $\mathfrak{f}_{J,\mathbf{I}}\mathfrak{f}_{\emptyset,J}$ by $\mathfrak{f}$ (see 36.4(b)). This (or rather its mixed analogue) gives rise to the following equality in $\mathfrak{K}(D)$:

$$\tilde{f}_J\tilde{e}_J\mathfrak{f}(x') = \sum_{u\in\mathbf{W}^J} v^{2m_u}\mathfrak{f}\rho_J\Phi_u(x')$$

for any $x' \in \mathfrak{K}(Z_{\emptyset,D})$, or equivalently

$$f_Je_J\mathfrak{f}(x') = \sum_{u\in\mathbf{W}^J} v^{2m_u-2\alpha_J}\mathfrak{f}\rho_J\Phi_u(x').$$

Taking $x' = \eta(x)$ where $x \in \mathfrak{K}(C)_n^D$ (see 41.7) and using 41.7(b) we obtain

$$\begin{aligned}&(v^2-1)^{2\mathbf{r}}f_Je_J\mathfrak{f}\eta(x)\\ &= \sum_{u\in\mathbf{W}^J}\sum_{\lambda\in\underline{\mathfrak{s}}_n} v^{-2l(u)}\mathfrak{f}\rho_J\eta([u^{-1};\lambda]' * x * [\epsilon_D(u); \underline{D}(u^{-1}\lambda)]')\end{aligned}$$

and using 42.1(i),

$$\begin{aligned}&(v^2-1)^{2\mathbf{r}}f_Je_J\mathfrak{f}\eta(x)\\ &= \sum_{u\in\mathbf{W}^J}\sum_{\lambda\in\underline{\mathfrak{s}}_n} v^{-2l(u)}\mathfrak{f}\eta\tilde{\rho}_J([u^{-1};\lambda]' * x * [\epsilon_D(u); \underline{D}(u^{-1}\lambda)]')\end{aligned}$$

for any $x \in \mathfrak{K}(C)_n^D$. Here we replace x by $\tilde{\omega}(y)$ where $y \in H_{n,D}$ and $\tilde{\rho}_J|_{\mathfrak{K}(C)_n^D}$ by $\tilde{\omega}|_{H_{n,D}}\rho_{J,n}\omega_{\mathfrak{K}(C)_n^D}$; using 40.7(b) we obtain:

$$\begin{aligned}f_Je_J\mathfrak{f}\eta\tilde{\omega}(y) &= \sum_{u\in\mathbf{W}^J}\sum_{\lambda\in\underline{\mathfrak{s}}_n}\mathfrak{f}\eta\tilde{\omega}\rho_{J,n}(\tilde{T}_{u^{-1}}1_\lambda y\tilde{T}_{\epsilon_D(u)}1_{\underline{D}(u^{-1}\lambda)})\\ &= \mathfrak{f}\eta\tilde{\omega}\rho_{J,n}(\sum_{u\in\mathbf{W}^J}\tilde{T}_{u^{-1}}y\tilde{T}_{\epsilon_D(u)}).\end{aligned}$$

The lemma is proved.

42.5. As in 34.12 let $\mathfrak{U}$ be the subfield of $\bar{\mathbf{Q}}_l$ generated by the roots of 1. Let $\Phi : H_n^D \to \mathcal{A} \otimes_{\mathbf{Z}} H_n^{D,\infty}$ be as in 34.12 (a special case of a definition in 34.1) and let $\Phi^1 : H_n^{D,1} \to \mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D,\infty}$ be the specialization of Φ for $v = 1$ (see 34.12(b)). Let $\tilde{\mathcal{A}} = \mathfrak{U}[v, v^{-1}]$, let $H_n^{D,\tilde{\mathcal{A}}} = \tilde{\mathcal{A}} \otimes \mathcal{A} H_n^D$ and let $\Phi^{\tilde{\mathcal{A}}} : H_n^{D,\tilde{\mathcal{A}}} \to \tilde{\mathcal{A}} \otimes_{\mathbf{Z}} H_n^{D,\infty}$ be the homomorphism obtained from Φ by extending the scalars from $\mathcal{A}$ to $\tilde{\mathcal{A}}$.

Let E be an $H_n^{D,1}$-module of finite dimension over $\mathfrak{U}$. Since Φ^1 is an isomorphism of $\mathfrak{U}$-algebras (see 34.12(b)) we may regard E as an $\mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D,\infty}$-module E^∞ via $(\Phi^1)^{-1}$. By extension of scalars, $\tilde{\mathcal{A}} \otimes_{\mathfrak{U}} E^\infty$ is naturally a module over

$\tilde{\mathcal{A}} \otimes_{\mathfrak{U}} (\mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D,iy}) = \tilde{\mathcal{A}} \otimes_{\mathbf{Z}} H_n^{D,iy}$

and this can be regarded as an $H_n^{D,\tilde{\mathcal{A}}}$-module $E^{\tilde{\mathcal{A}}}$ via $\Phi^{\tilde{\mathcal{A}}}$.

Let $J \subset \mathbf{I}$ be such that $\epsilon(J) = J$. Let $H_{J,n}^D$ be the $\mathcal{A}$-algebra of H_n^D generated by $1_\lambda, \lambda \in \underline{\mathfrak{s}}_n$ by $\tilde{T}_w, w \in \mathbf{W}_J$ and by $\tilde{T}_{\underline{D}}$. Note that $\{\tilde{T}_{w\underline{D}'} 1_\lambda; w \in \mathbf{W}_J, \underline{D}' =$ power of $\underline{D}\}$ is an $\mathcal{A}$-basis of $H_{J,n}^D$. Let $H_{J,n}^{D,1} = \mathfrak{U} \otimes_{\mathcal{A}} H_{J,n}^D$ where $\mathfrak{U}$ is regarded as an $\mathcal{A}$-algebra via $v \mapsto 1$. Let $H_{J,n}^{D,\tilde{\mathcal{A}}} = \tilde{\mathcal{A}} \otimes_{\mathcal{A}} H_{J,n}^D$. Note that $H_{J,n}^{D,\tilde{\mathcal{A}}}$ is naturally a subalgebra of $H_n^{D,\tilde{\mathcal{A}}}$. Hence $E^{\tilde{\mathcal{A}}}$ may be regarded as an $H_{J,n}^{D,\tilde{\mathcal{A}}}$-module $(E^{\tilde{\mathcal{A}}})_J$. This $H_{J,n}^{D,\tilde{\mathcal{A}}}$-module may be induced to an $H_n^{D,\tilde{\mathcal{A}}}$-module $\mathrm{IND}((E^{\tilde{\mathcal{A}}})_J) := H_n^{D,\tilde{\mathcal{A}}} \otimes_{H_{J,n}^{D,\tilde{\mathcal{A}}}} E_J^{\tilde{\mathcal{A}}}$.

Next, $H_{J,n}^{D,1}$ is naturally a subalgebra of $H_n^{D,1}$. Hence E may be regarded as an $H_{J,n}^{D,1}$-module E_J. This $H_{J,n}^{D,1}$-module may be induced to an $H_n^{D,1}$-module $\mathrm{ind}(E_J) := H_n^{D,1} \otimes_{H_{J,n}^{D,1}} E_J$. Define an $H_{J,n}^{D,\tilde{\mathcal{A}}}$-module $(\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}$ in terms of $\mathrm{ind}(E_J)$ in the same way as $E^{\tilde{\mathcal{A}}}$ was defined in terms of E. By extension of scalars from $\tilde{\mathcal{A}}$ to $\mathfrak{U}(v)$ (the quotient field of $\tilde{\mathcal{A}}$), $\mathrm{IND}((E^{\tilde{\mathcal{A}}})_J)$, $(\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}$ give rise to $\mathfrak{U}(v) \otimes_{\mathcal{A}} H_n^D$-modules $\mathfrak{U}(v) \otimes_{\tilde{\mathcal{A}}} \mathrm{IND}((E^{\tilde{\mathcal{A}}})_J)$, $\mathfrak{U}(v) \otimes_{\tilde{\mathcal{A}}} (\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}$. We show:

(a) *These two $\mathfrak{U}(v) \otimes_{\mathcal{A}} H_n^D$-modules are isomorphic.*

Since $\mathfrak{U}(v) \otimes_{\mathcal{A}} H_n^D$, $H_n^{D,1}$ are (finite dimensional) semisimple algebras (see 34.12) it follows by standard arguments that it is enough to show that $\mathrm{IND}((E^{\tilde{\mathcal{A}}})_J)$, $(\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}$ become isomorphic $H_n^{D,1}$-modules under the specialization $v = 1$. First we note that under the specialization $v = 1$, $E^{\tilde{\mathcal{A}}}$ becomes the $H_n^{D,1}$-module E. (This is because the specialization of $\Phi^{\tilde{\mathcal{A}}}$ at $v = 1$ cancels $(\Phi_1)^{-1}$.) In particular, the specialization of $(\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}$ for $v = 1$ is $\mathrm{ind}(E_J)$. Moreover, from the definition of induction, the specialization of $\mathrm{IND}((E^{\tilde{\mathcal{A}}})_J)$ for $v = 1$ is the same as $\mathrm{ind}(E'_J)$ where E' is the specialization of $E^{\tilde{\mathcal{A}}}$ for $v = 1$ that is, $E' = E$. This proves (a).

Lemma 42.6. *We preserve the setup of 42.5. Let $\mathfrak{U}(v) \otimes_{\tilde{\mathcal{A}}} E^{\tilde{\mathcal{A}}}$, $\mathfrak{U}(v) \otimes_{\tilde{\mathcal{A}}} (\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}$ be the $\mathfrak{U}(v) \otimes_{\mathcal{A}} H_n^D$-module obtained from $E^{\tilde{\mathcal{A}}}$, $(\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}$ by extension of scalars from $\tilde{\mathcal{A}}$ to $\mathfrak{U}(v)$. Let $y \in H_{n,D}$. We have:*

$\mathrm{tr}(\delta_J(y)\tilde{T}_{\underline{D}}, \mathfrak{U}(v) \otimes_{\tilde{\mathcal{A}}} E^{\tilde{\mathcal{A}}}) = \mathrm{tr}(y\tilde{T}_{\underline{D}}, \mathfrak{U}(v) \otimes_{\tilde{\mathcal{A}}} (\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}})$.

Let $H_{J,n}$ be the $\mathcal{A}$-subalgebra of H_n defined in 31.8. Define an $\mathcal{A}$-linear map $p_J : H_n \to H_{J,n}$ by $p_J(\tilde{T}_z 1_\lambda)) = \tilde{T}_z 1_\lambda$ if $z \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}_n$, $p_J(\tilde{T}_z 1_\lambda) = 0$ if $z \in \mathbf{W} - \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}_n$. We show that

(a) $p_J(\tilde{T}_u h') = \delta_{u,1} h'$ if $u \in \mathbf{W}^J, h' \in H_{J,n}$.

We may assume that $h' = \tilde{T}_b 1_\lambda, b \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}_n$. Then $p_J(\tilde{T}_u \tilde{T}_b 1_\lambda) = p_J(\tilde{T}_{ub} 1_\lambda) = \delta_{u,1} \tilde{T}_{ub} 1_\lambda = \delta_{u,1} \tilde{T}_b 1_\lambda$, as required.

We show:

(b) $p_J(hh') = p_J(h)h'$ for any $h \in H_n, h' \in H_{J,n}$.

We may assume $h = \tilde{T}_u \tilde{T}_b 1_\nu, h' = \tilde{T}_a 1_\lambda$, $u \in \mathbf{W}^J, a, b \in \mathbf{W}_J$, $\lambda, \nu \in \underline{\mathfrak{s}}_n$. We must show tha $p_J(\tilde{T}_u \tilde{T}_b 1_\nu \tilde{T}_a 1_\lambda) = p_J(\tilde{T}_u \tilde{T}_b 1_\nu) \tilde{T}_a 1_\lambda$. If $u \ne 1$, both sides are zero by (a). If $u = 1$ both sides are $\tilde{T}_b 1_\nu \tilde{T}_a 1_\lambda$. This proves (b).

By 34.13(a) we have

(c) $p_\emptyset(\tilde{T}_x \tilde{T}_{x'} 1_\lambda) = \delta_{xx',1}$ for $x, x' \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n$.

For $u, u' \in \mathbf{W}^J, \lambda \in \underline{\mathfrak{s}}_n$ we write $\tilde{T}_{u^{-1}} \tilde{T}_{u'} 1_\lambda = \sum_{a \in \mathbf{W}} f_a \tilde{T}_a 1_\lambda$ where $f_a \in \mathcal{A}$. For $a' \in \mathbf{W}_J$ we have

$$\tilde{T}_{a'^{-1} u^{-1}} \tilde{T}_{u'} 1_\lambda = \tilde{T}_{a'^{-1}} \tilde{T}_{u^{-1}} \tilde{T}_{u'} 1_\lambda = \sum_{a \in \mathbf{W}} f_a \tilde{T}_{a'^{-1}} \tilde{T}_a 1_\lambda.$$

Applying $p_\emptyset$ to this and using (c) gives $f_{a'} = \delta_{u',ua'} = \delta_{a',1} \delta_{u,u'}$ so that

$p_J(\tilde{T}_{u^{-1}} \tilde{T}_{u'} 1_\lambda) = \sum_{a \in \mathbf{W}_J} f_a \tilde{T}_a 1_\lambda = \delta_{u,u'} \tilde{T}_1 1_\lambda$.

Since this holds for any $\lambda \in \underline{\mathfrak{s}}_n$ we have

(d) $p_J(\tilde{T}_{u^{-1}} \tilde{T}_{u'}) = \delta_{u,u'} \tilde{T}_1$.

Let $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n, u \in \mathbf{W}^J$. We have

$\tilde{T}_w 1_\lambda \tilde{T}_u = \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,u,u',a,\lambda} \tilde{T}_{u'} \tilde{T}_a 1_{u^{-1}\lambda}$

where $c_{w,u,u',a,\lambda} \in \mathcal{A}$ are uniquely determined. It follows that

$$\tilde{T}_{u^{-1}} \tilde{T}_w 1_\lambda \tilde{T}_{\epsilon(u)} = \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u',a,\lambda} \tilde{T}_{u^{-1}} \tilde{T}_{u'} \tilde{T}_a 1_{\epsilon(u)^{-1}\lambda}.$$

Applying p_J and using (b),(d) we obtain

$$p_J(\tilde{T}_{u^{-1}} \tilde{T}_w 1_\lambda \tilde{T}_{\epsilon(u)}) = \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u',a,\lambda} p_J(\tilde{T}_{u^{-1}} \tilde{T}_{u'}) \tilde{T}_a 1_{\epsilon(u)^{-1}\lambda}$$

(e)

$$= \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u',a,\lambda} \delta_{u,u'} \tilde{T}_a 1_{\epsilon(u)^{-1}\lambda} = \sum_{a \in \mathbf{W}_J} c_{w,\epsilon(u),u,a,\lambda} \tilde{T}_a 1_{\epsilon(u)^{-1}\lambda}.$$

Let $(e_i)_{i \in X}$ be a basis of the free $\tilde{\mathcal{A}}$-module $E^{\tilde{\mathcal{A}}}$. For $a \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}_n$ we have $\tilde{T}_a 1_\lambda \tilde{T}_{\underline{D}} e_i = \sum_{i' \in X} \tilde{c}_{a,\lambda,i,i'} e_{i'}$ where $\tilde{c}_{a,\lambda,i,i'} \in \tilde{\mathcal{A}}$.

Since $H_n^{D,\tilde{\mathcal{A}}}$ is a free right $H_{J,n}^{D,\tilde{\mathcal{A}}}$-module with basis $\{\tilde{T}_u; u \in \mathbf{W}^J\}$ we see that $\{\tilde{T}_u \otimes e_i; u \in \mathbf{W}^J, i \in X\}$ is a basis of the free $\tilde{\mathcal{A}}$-module $\mathrm{ind}((E^{\tilde{\mathcal{A}}})_J)$.

Let $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n, u \in \mathbf{W}^J$ be such that $w\underline{D}\lambda = \lambda$. In $\mathrm{IND}((E^{\tilde{\mathcal{A}}})_J)$ we have

$$\begin{aligned}
&\tilde{T}_w 1_\lambda \tilde{T}_{\underline{D}}(\tilde{T}_u \otimes e_i) = (\tilde{T}_w 1_\lambda \tilde{T}_{\epsilon(u)} \tilde{T}_{\underline{D}}) \otimes e_i \\
&= \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u',a,\lambda} (\tilde{T}_{u'} \tilde{T}_a 1_{\epsilon(u)^{-1}\lambda} \tilde{T}_{\underline{D}}) \otimes e_i \\
&= \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u',a,\lambda} \tilde{T}_{u'} \otimes (\tilde{T}_a 1_{\epsilon(u)^{-1}\lambda} \tilde{T}_{\underline{D}} e_i) \\
&= \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J, i' \in X} c_{w,\epsilon(u),u',a,\lambda} \tilde{c}_{a,\epsilon(u)^{-1}\lambda,i,i'} \tilde{T}_{u'} \otimes e_{i'}.
\end{aligned}$$

Hence, using (e),

$$\begin{aligned}
&\mathrm{tr}(\tilde{T}_w 1_\lambda \tilde{T}_{\underline{D}}, \mathrm{IND}((E^{\tilde{\mathcal{A}}})_J)) = \sum_{u \in \mathbf{W}^J, a \in \mathbf{W}_J, i \in X} c_{w,\epsilon(u),u,a,\lambda} \tilde{c}_{a,\epsilon(u)^{-1}\lambda,i,i} \\
&= \sum_{u \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u,a,\lambda} \mathrm{tr}(\tilde{T}_a 1_{\epsilon(u)^{-1}\lambda} \tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) \\
&= \sum_{u \in \mathbf{W}^J} \mathrm{tr}(\sum_{a \in \mathbf{W}_J} c_{w,\epsilon(u),u,a,\lambda} \tilde{T}_a 1_{\epsilon(u)^{-1}\lambda} \tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) \\
&= \mathrm{tr}(p_J(\sum_{u \in \mathbf{W}^J} \tilde{T}_{u^{-1}} \tilde{T}_w 1_\lambda \tilde{T}_{\epsilon(u)}) \tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) \\
&= \mathrm{tr}(\rho_{J,n}(\sum_{u \in \mathbf{W}^J} \tilde{T}_{u^{-1}} \tilde{T}_w 1_\lambda \tilde{T}_{\epsilon(u)}) \tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) = \mathrm{tr}(\delta_J(\tilde{T}_w 1_\lambda) \tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}).
\end{aligned}$$

Thus we have

$$\mathrm{tr}(\delta_J(\tilde{T}_w 1_\lambda) \tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) = \mathrm{tr}(\tilde{T}_w 1_\lambda \tilde{T}_{\underline{D}}, \mathrm{IND}((E^{\tilde{\mathcal{A}}})_J)) = \mathrm{tr}(\tilde{T}_w 1_\lambda \tilde{T}_{\underline{D}}, (\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}})$$

where the second equality follows from 42.5(a). Since the elements $\tilde{T}_w 1_\lambda$ as above generate the $\mathcal{A}$-module $H_{n,D}$, the lemma follows.

42.7. Let $\mathcal{V}$ be the $\mathbf{Q}$-vector subspace of $\mathbf{Q} \otimes \mathrm{Hom}(\mathbf{k}^*, \mathbf{T})$ spanned by the coroots. Let $\mathcal{V}_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V}$. The kernels of the roots $\mathcal{V}_{\mathbf{R}} \to \mathbf{R}$ a hyperplane arrangement which defines a partition of $\mathcal{V}_{\mathbf{R}}$ into facets in a standard way. Let $\mathcal{F}$ be the set of facets. Now the orbits of $\mathbf{W}$ on $\mathcal{F}$ are naturally indexed by the various subsets J of $\mathbf{I}$. This gives a partition $\mathcal{F} = \sqcup_{J \subset \mathbf{I}} \mathcal{F}_J$. For example $\mathcal{F}_\emptyset$ consists of all Weyl chambers. If $F \in \mathcal{F}_J$ then F is homeomorphic to a real affine space of dimension $|\mathbf{I} - J|$ hence we have $H^i_c(F) = 0$ if $i \neq |\mathbf{I} - J|$ and $H^{|\mathbf{I}-J|}_c(F) = \Lambda^{|\mathbf{I}-J|}[F]$; here we write $H^i_c(?)$ instead of $H^i_c(?, \mathbf{R})$, $[F]$ denotes the vector subspace of $\mathcal{V}_{\mathbf{R}}$ in which F is open dense and $\Lambda^{|\mathbf{I}-J|}[F]$ is the top exterior power of $[F]$. Note that $[F] = \mathbf{R} \otimes_{\mathbf{Q}} ([F]_{\mathbf{Q}})$ for a well defined $\mathbf{Q}$-subspace $[F]_{\mathbf{Q}}$ of $\mathcal{V}$. For any $\underline{D}$-orbit η on the set of subsets of $\mathbf{I}$ let $\mathcal{V}^\eta_{\mathbf{R}} = \cup_{J \in \eta} \cup_{F \in \mathcal{F}_J} F \subset \mathcal{V}_{\mathbf{R}}$ and let $r_\eta = |\mathbf{I} - J|$ for some/any $J \in \eta$. We have $H^i_c(\mathcal{V}^\eta_{\mathbf{R}}) = 0$ if $i \neq r_\eta$, $H^{r_\eta}_c(\mathcal{V}^\eta_{\mathbf{R}}) = \oplus_{J \in \eta} \oplus_{F \in \mathcal{F}_J} \Lambda^{r_\eta}[F]$. Note also that $H^i_c(\mathcal{V}_{\mathbf{R}}) = 0$ if $i \neq |\mathbf{I}|$ and $H^{|\mathbf{I}|}_c(\mathcal{V}_{\mathbf{R}}) = \Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathbf{R}}$. The $\mathbf{W}^D$-action on

$\mathbf{T}$ induces a linear action of $\mathbf{W}^D$ on $\mathcal{V}_{\mathbf{R}}$. This action restricts for any η to a $\mathbf{W}^D$-action on $\mathcal{V}_{\mathbf{R}}^{\eta}$ and this induces a $\mathbf{W}^D$-action on $H_c^{r_\eta}(\mathcal{V}_{\mathbf{R}}^{\eta})$. It also induces a natural $\mathbf{W}^D$-action on $H_c^{|\mathbf{I}|}(\mathcal{V}_{\mathbf{R}}) = \Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathbf{R}}$. The long cohomology exact sequences attached to the partition $\mathcal{V}_{\mathbf{R}} = \cup_\eta \mathcal{V}_{\mathbf{R}}^{\eta}$ show that $(-1)^{|\mathbf{I}|}H_c^{|\mathbf{I}|}(\mathcal{V}_{\mathbf{R}}) = \sum_\eta (-1)^{r_\eta} H_c^{r_\eta}(\mathcal{V}_{\mathbf{R}}^{\eta})$ in the Grothendieck group of representations of $\mathbf{W}^D$ over $\mathbf{R}$ that is,

$$\begin{aligned}&\Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathbf{R}} \oplus \oplus_{\eta; r_\eta = |\mathbf{I}|+1 \mod 2}(\oplus_{J\in\eta} \oplus_{F\in\mathcal{F}_J} \Lambda^{r_\eta}[F])\\ &\cong \oplus_{\eta; r_\eta=|\mathbf{I}| \mod 2}(\oplus_{J\in\eta}\oplus_{F\in\mathcal{F}_J} \Lambda^{r_\eta}[F])\end{aligned}$$

as representations of $\mathbf{W}^D$ over $\mathbf{R}$. All real representations in this formula come naturally from representations of $\mathbf{W}^D$ over $\mathbf{Q}$. Hence the previous formula remains valid (as representations of $\mathbf{W}^D$ over $\mathbf{Q}$) if $\mathcal{V}_{\mathbf{R}}, [F]$ are replaced by $\mathcal{V}, [F]_{\mathbf{Q}}$ and the exterior powers are taken over $\mathbf{Q}$. Tensoring both sides (over $\mathbf{Q}$) by $\mathfrak{U}$ (as in 42.5) we obtain

$$\begin{aligned}&\Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}} \oplus \oplus_{\eta; r_\eta = |\mathbf{I}|+1 \mod 2}(\oplus_{J\in\eta} \oplus_{F\in\mathcal{F}_J} \Lambda^{r_\eta}[F]_{\mathfrak{U}})\\ &\cong \oplus_{\eta; r_\eta=|\mathbf{I}| \mod 2}(\oplus_{J\in\eta}\oplus_{F\in\mathcal{F}_J} \Lambda^{r_\eta}[F]_{\mathfrak{U}})\end{aligned} \tag{a}$$

as representations of $\mathbf{W}^D$ over $\mathfrak{U}$; here $\mathcal{V}_{\mathfrak{U}} = \mathfrak{U}\otimes_{\mathbf{Q}} \mathcal{V}$, $[F]_{\mathfrak{U}} = \mathfrak{U}\otimes_{\mathbf{Q}} [F]_{\mathbf{Q}}$ and the exterior powers are taken over $\mathfrak{U}$. We may view (a) as an isomorphism of $H_n^{D,1}$-modules: the $\mathbf{W}^D$-modules in (a) may be viewed as $H_n^{D,1}$-modules via the algebra homomorphism $H_n^{D,1} \longrightarrow \mathfrak{U}[\mathbf{W}^D]$ given by $\tilde{T}_w \mapsto w$ for $w\in\mathbf{W}^D$, $1_\lambda \mapsto 0$ for $\lambda\ne\lambda_0$, $1_{\lambda_0}\mapsto 1$ (here λ_0 is the neutral element of the abelian group $\underline{\mathfrak{s}}_n$, see 28.1).

We define an $\mathfrak{U}$-linear map $\Delta : H_n^{D,1} \longrightarrow H_n^{D,1}\otimes H_n^{D,1}$ by $\Delta(\tilde{T}_w) = \tilde{T}_w\otimes\tilde{T}_w$ for $w\in\mathbf{W}^D$ and $\Delta(1_\lambda) = \sum_{\lambda_1,\lambda_2\in\underline{\mathfrak{s}}_n;\lambda_1\lambda_2=\lambda} 1_{\lambda_1}\otimes 1_{\lambda_2}$ for $\lambda\in\underline{\mathfrak{s}}_n$. (Here we use the abelian group structure on $\underline{\mathfrak{s}}_n$, a subgroup of $\underline{\mathfrak{s}}$, see 28.1.) This makes $H_n^{D,1}$ into a Hopf algebra. (Note that the analogous formulas do not make H_n^D into a Hopf algebra.) It follows that for any two $H_n^{D,1}$-modules E_1, E_2, the $\mathfrak{U}$-vector space $E_1\otimes E_2$ is naturally an $H_n^{D,1}$-module.

Now let E be an $H_n^{D,1}$-module of finite dimension over $\mathfrak{U}$. Then we can take tensor product of each $H_n^{D,1}$-module in (a) with E and we obtain an isomorphism of $H_n^{D,1}$-modules

$$E\otimes\Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}} \oplus \oplus_{\eta;r_\eta=|\mathbf{I}|+1 \mod 2} X_\eta \cong \oplus_{\eta;r_\eta=|\mathbf{I}| \mod 2} X_\eta$$

where $X_\eta = E\otimes\oplus_{J\in\eta}\oplus_{F\in\mathcal{F}_J}\Lambda^{r_\eta}[F]_{\mathfrak{U}}$. Applying to this the functor $E\mapsto E^{\tilde{\mathcal{A}}}$, see 42.5, we deduce an isomorphism of $H_n^{D,\tilde{\mathcal{A}}}$-modules

$$(E\otimes\Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}})^{\tilde{\mathcal{A}}} \oplus \oplus_{\eta;r_\eta=|\mathbf{I}|+1 \mod 2} X_\eta^{\tilde{\mathcal{A}}} \cong \oplus_{\eta;r_\eta=|\mathbf{I}| \mod 2} X_\eta^{\tilde{\mathcal{A}}}.$$

We deduce that for $y\in H_{n,D}$ we have

$$\mathrm{tr}(y\tilde{T}_{\underline{D}}, (E\otimes\Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}})^{\tilde{\mathcal{A}}}) = \sum_\eta (-1)^{r_\eta+|\mathbf{I}|}\mathrm{tr}(y\tilde{T}_{\underline{D}}, X_\eta^{\tilde{\mathcal{A}}}). \tag{b}$$

We have $X_\eta=\oplus_{J\in\eta}X^J$ where $X^J=E\otimes(\oplus_{F\in\mathcal{F}_J}\Lambda^{r_\eta}[F]_{\mathfrak{U}})$.

Assume first that η consists of at least two subsets of $\mathbf{I}$. Then each X_J is stable under $H_n^{D,1}$ and is mapped by $\tilde{T}_{\underline{D}}$ into $X_{J'}$ with $J\ne J'$. From the definitions we have $X_\eta^{\tilde{\mathcal{A}}}=\oplus_{J\in\eta}\tilde{\mathcal{A}}\otimes_{\mathfrak{U}}X_J$ as an $\tilde{\mathcal{A}}$-module and each summand $\tilde{\mathcal{A}}\otimes_{\mathfrak{U}}X_J$ is stable under H_n and is mapped by $\tilde{T}_{\underline{D}}$ into a summand $\tilde{\mathcal{A}}\otimes_{\mathfrak{U}}X_{J'}$ with $J\ne J'$. It follows that for our η we have

(c) $\mathrm{tr}(y\tilde{T}_{\underline{D}},X_\eta^{\tilde{\mathcal{A}}})=0$.

Next assume that η consists of a single subset J of $\mathbf{I}$. We have $\underline{D}(J)=J$. Let F_J be the unique facet in $\mathcal{F}_J$ such that F_J is contained in the closure of the dominant Weyl chamber. Then F_J is stable under the the subgroup $\mathbf{W}_J^D$ of $\mathbf{W}^D$ generated by $\mathbf{W}_J$ and $\underline{D}$ and X_η may be identified with $E\otimes(H_n^{D,1}\otimes_{H_{J,n}^{D,1}}(\Lambda^{|\mathbf{I}-J|}[F_J]_{\mathfrak{U}}))$. Here $\Lambda^{|\mathbf{I}-J|}[F_J]_{\mathfrak{U}}$ is regarded as a WW_J^D-module and then is viewed as a $H_{J,n}^{D,1}$-module via the canonical algebra homomorphism $H_{J,n}^{D,1}\to\mathfrak{U}[\mathbf{W}_J^D]$; thus 1_λ acts on it as 1 if $\lambda=\lambda_0$ and as 0 if $\lambda\ne\lambda_0$. Note that in the $\mathbf{W}_J^D$-module $\Lambda^{|\mathbf{I}-J|}[F_J]_{\mathfrak{U}}$, $\mathbf{W}_J$ acts trivially (since $\mathbf{W}_J$ acts trivially on $[F_J]_{\mathfrak{U}}$) and $\underline{D}$ acts as multiplication by $(-1)^{|\mathbf{I}-J|-|(\mathbf{I}-J)_\epsilon|}$). Let $X'_\eta=E\otimes(H_n^{D,1}\otimes_{H_{J,n}^{D,1}}\mathfrak{U})$ where $\mathfrak{U}$ is regarded as a $H_{J,n}^{D,1}$-module coming from the trivial representation of $\mathbf{W}_J^D$. We see that we may identify X_η,X'_η in a way compatible with the H_n^1-module structures and so that the action of $\tilde{T}_{\underline{D}}$ on X_η corresponds to $(-1)^{|\mathbf{I}-J|-|(\mathbf{I}-J)_\epsilon|}$ times the action of $\tilde{T}_{\underline{D}}$ on X'_η. Using the definitions we see that we may identify $X_\eta^{\tilde{\mathcal{A}}},X_\eta'^{\tilde{\mathcal{A}}}$ in a way compatible with the H_n-module structures and so that the action of $\tilde{T}_{\underline{D}}$ on $X_\eta^{\tilde{\mathcal{A}}}$ corresponds to $(-1)^{|\mathbf{I}-J|-|(\mathbf{I}-J)_\epsilon|}$ times the action of $\tilde{T}_{\underline{D}}$ on $X_\eta'^{\tilde{\mathcal{A}}}$. From the definitions we have $X'_\eta=\mathrm{ind}(E_J)$ (notation of 42.5). We see that for our η we have

(d) $$\mathrm{tr}(y\tilde{T}_{\underline{D}},X_\eta^{\tilde{\mathcal{A}}})=(-1)^{|\mathbf{I}-J|-|(\mathbf{I}-J)_\epsilon|}\mathrm{tr}(y\tilde{T}_{\underline{D}},(\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}).$$

From the definitions (34.4) we see that there is a unique $\tilde{\mathcal{A}}$-algebra homomorphism $\vartheta:H_n^{D,\tilde{\mathcal{A}}}D\to H_n^{D,\tilde{\mathcal{A}}}$ such that

$\vartheta(1_\lambda)=1_\lambda$ for any $\lambda\in\underline{\mathfrak{s}}_n$,

$\vartheta(\tilde{T}_w)=(-1)^{l(w)}\tilde{T}_{w^{-1}}^{-1}$ for any $w\in\mathbf{W}$,

$\vartheta(\tilde{T}_{\underline{D}})=(-1)^{|\mathbf{I}|-|\mathbf{I}_\epsilon|}\tilde{T}_{\underline{D}}$.

We have $\vartheta^2=1$.

Using ϑ and $E^{\tilde{\mathcal{A}}}$ we can define a new $H_n^{D,\tilde{\mathcal{A}}}$-module $E^{\tilde{\mathcal{A}},\vartheta}$ with the same underlying $\tilde{\mathcal{A}}$-module as $E^{\tilde{\mathcal{A}}}$ but with $x\in H_n^{D,\tilde{\mathcal{A}}}$ acting on $E^{\tilde{\mathcal{A}},\vartheta}$ in the same way that $\vartheta(x)$ acts on $E^{\tilde{\mathcal{A}}}$. We show:

(e) *under extension of scalars from $\tilde{\mathcal{A}}$ to $\mathfrak{U}(v)$, the $H_n^{D,\tilde{\mathcal{A}}}$-modules $E^{\tilde{\mathcal{A}},\vartheta}$ and $(E\otimes\Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}})^{\tilde{\mathcal{A}}}$ become isomorphic $\mathfrak{U}(v)\otimes_{\mathcal{A}}H_n^D$-modules.*

As in the proof of 42.5(a) it is enough to show that these $H_n^{D,\tilde{\mathcal{A}}}$-modules become isomorphic $H_n^{D,1}$-modules under the specialization $v=1$. Thus it is enough to

show that $E^{\tilde{\mathcal{A}},\vartheta}|_{v=1} \cong E\otimes\Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}}$ as $H_n^{D,1}$-modules. Now the underlying $\mathfrak{U}$-vector space of $E^{\tilde{\mathcal{A}},\vartheta}|_{v=1}$ is E but the action of $x\in H_n^{D,1}$ on $E^{\tilde{\mathcal{A}},\vartheta}|_{v=1}$ is the same as the action of $\vartheta_1(x)$ on E. Here $\vartheta_1 : H_n^{D,1}\to H_n^{D,1}$ is the specialization of ϑ_1 for $v=1$. Note that $\vartheta_1(1_\lambda)=1_\lambda$ for any $\lambda\in\underline{\mathfrak{s}}_n$ and $\vartheta_1(\tilde{T}_w)=\gamma_w\tilde{T}_w$ for any $w\in\mathbf{W}^D$, where $\gamma_w=\pm1$ is the scalar by which w acts in the $\mathbf{W}^D$-module $\Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}}$. The desired result follows.

Combining (b),(c),(d),(e) we see that for any $y\in H_{n,D}$ we have

$$(-1)^{|\mathbf{I}|+|\mathbf{I}_\epsilon|}\mathrm{tr}(\vartheta(y\tilde{T}_{\underline{D}}),E^{\tilde{\mathcal{A}}})=\sum_{J\subset\mathbf{I};\epsilon(J)=J}(-1)^{|J_\epsilon|}\mathrm{tr}(y\tilde{T}_{\underline{D}},(\mathrm{ind}(E_J))^{\tilde{\mathcal{A}}}).$$

Replacing here $(-1)^{|\mathbf{I}|+|\mathbf{I}_\epsilon|}\vartheta(y\tilde{T}_{\underline{D}})$ by $\vartheta(y)\tilde{T}_{\underline{D}}$ and using Lemma 42.6 we may rewrite this as follows:

$$\mathrm{tr}(\vartheta(y)\tilde{T}_{\underline{D}},E^{\tilde{\mathcal{A}}})=\sum_{J\subset\mathbf{I};\epsilon(J)=J}(-1)^{|J_\epsilon|}\mathrm{tr}(\delta_J(y)\tilde{T}_{\underline{D}},E^{\tilde{\mathcal{A}}})$$

or equivalently (see 42.4) $\mathrm{tr}(\vartheta(y)\tilde{T}_{\underline{D}},E^{\tilde{\mathcal{A}}})=\mathrm{tr}(\delta(y)\tilde{T}_{\underline{D}},E^{\tilde{\mathcal{A}}})$. Since any simple $\mathfrak{U}(v)\otimes_{\mathcal{A}}H_n^D$-module can be obtained by extension of scalars (from $\tilde{\mathcal{A}}$ to $\mathfrak{U}(v)$) from some $E^{\tilde{\mathcal{A}}}$ as above, we deduce that

$$\mathrm{tr}((\delta(y)-\vartheta(y))\tilde{T}_{\underline{D}},\mathbf{E})=0$$

for any simple $\mathfrak{U}(v)\otimes_{\mathcal{A}}H_n^D$-module $\mathbf{E}$. Since $\mathfrak{U}(v)\otimes_{\mathcal{A}}H_n^D$ is a semisimple algebra, it follows that $(\delta(y)-\vartheta(y))\tilde{T}_{\underline{D}}$ belongs to the $\mathfrak{U}(v)$-subspace of $\mathfrak{U}(v)\otimes_{\mathcal{A}}H_n^D$ spanned by commutators $xx'-x'x$ with $x,x'\in\mathfrak{U}(v)\otimes_{\mathcal{A}}H_n^D$. Hence we have

$$g(\delta(y)-\vartheta(y))\tilde{T}_{\underline{D}}=\sum_{i=1}^m g_i(x_i\tilde{T}_{\underline{D}}^{s_i}x'_i\tilde{T}_{\underline{D}}^{1-s_i}-x'_i\tilde{T}_{\underline{D}}^{1-s_i}x_i\tilde{T}_{\underline{D}}^{s_i})$$

with $g\in\mathcal{A}-\{0\},g_i\in\mathcal{A},x_i\in H_n,x'_i\in H_n,s_i\in\mathbf{Z}$ that is,

(f) $$g(\delta(y)-vt(y))=\sum_{i=1}^m g_i(x_i\tilde{T}_{\underline{D}}^{s_i}x'_i\tilde{T}_{\underline{D}}^{-s_i}-x'_i\tilde{T}_{\underline{D}}^{1-s_i}x_i\tilde{T}_{\underline{D}}^{s_i-1}).$$

42.8. We show that for any $y,y'\in H_n$ we have

(a) $\mathfrak{f}\eta\tilde{\omega}(yy'-y'\tilde{T}_{\underline{D}}y\tilde{T}_{\underline{D}}^{-1})=0.$

Let $w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n$. Let $\mathcal{L}\in\lambda$. If $w\underline{D}\lambda=\lambda$, using notation and results in 31.6, 31.7 we have

$$\begin{aligned}&\mathfrak{f}\eta\tilde{\omega}(v^{l(w)}\tilde{T}_w1_{\underline{D}\lambda}))=gr(K_{\mathbf{I},D}^{w,\mathcal{L}}))\\&=\sum_A\chi_v^A(K_{\mathbf{I},D}^{w,\mathcal{L}}))=\sum_A\tilde{\zeta}^A(v^{l(w)}\tilde{T}_w1_{\underline{D}\lambda}\tilde{T}_{\underline{D}})=\sum_A\zeta^A(v^{l(w)}\tilde{T}_w1_{\underline{D}\lambda}\tilde{T}_{\underline{D}})\end{aligned}$$

(the last equation comes from 31.7(e); A runs over the objects in $\hat{D}$ up to isomorphism such that $\zeta^A \ne 0$.) The equation

$\mathfrak{f}\eta\tilde{\omega}(v^{l(w)}\tilde{T}_w 1_{\underline{D}}\lambda)) = \sum_A \zeta^A(v^{l(w)}\tilde{T}_w 1_{\underline{D}}\lambda\tilde{T}_{\underline{D}})$

holds also if $w\underline{D}\lambda \ne \lambda$ (in this case both sides are 0). It follows that

$\mathfrak{f}\eta\tilde{\omega}(x)) = \sum_A \zeta^A(x\tilde{T}_{\underline{D}})$ for any $x \in H_n$.

We deduce

$\mathfrak{f}\eta\tilde{\omega}(yy' - y'\tilde{T}_{\underline{D}}y\tilde{T}_{\underline{D}}^{-1}) = \sum_A(\zeta^A(yy'\tilde{T}_{\underline{D}}) - \zeta^A(y\epsilon(y)\tilde{T}_{\underline{D}}) = 0$

where the last equality follows from 31.8. This proves (a).

Proposition 42.9. *Let $y \in H$. We have $\mathbf{d}(\mathfrak{f}\eta\tilde{\omega}(y)) = \mathfrak{f}\eta\tilde{\omega}(\vartheta(y)) \in \mathfrak{K}(D)$ with $\mathbf{d}: \mathfrak{K}(D) \to \mathfrak{K}(\Delta)$ as in 42.2.*

If $y \in H'_D$ (see 42.3), both sides of the desired equality are 0. (Note that ϑ maps H_D into itself and H'_D into itself.) Hence we may assume that $y \in H_D$. We can assume that $y \in H_n$ where $n \in \mathbf{N}^*_{\mathbf{k}}$. Then $y \in H_{n,D}$. By 42.4 it is enough to show that $\mathfrak{f}\eta\tilde{\omega}(\delta(y) - \vartheta(y)) = 0$. Let g, g_i, x_i, x'_i, s_i be as in 42.7(f). Since $g \ne 0$ it is enough to show that $g\mathfrak{f}\eta\tilde{\omega}(\delta(y) - \vartheta(y)) = 0$ or that $\mathfrak{f}\eta\tilde{\omega}(g(\delta(y) - \vartheta(y))) = 0$. Using 42.7 it is enough to show that

$$\mathfrak{f}\eta\tilde{\omega}(\sum_{i=1}^m g_i(x_i\tilde{T}_{\underline{D}}^{s_i}x'_i\tilde{T}_{\underline{D}}^{-s_i} - x'_i\tilde{T}_{\underline{D}}^{1-s_i}x_i\tilde{T}_{\underline{D}}^{s_i-1}) = 0.$$

Hence it is enough to show that

$\mathfrak{f}\eta\tilde{\omega}(x\tilde{T}_{\underline{D}}^s x'\tilde{T}_{\underline{D}}^{-s} - x'\tilde{T}_{\underline{D}}^{1-s}x\tilde{T}_{\underline{D}}^{s-1}) = 0$

for any $x, x' \in H_n$ and any $s \in \mathbf{Z}$. We have

$$x\tilde{T}_{\underline{D}}^s x'\tilde{T}_{\underline{D}}^{-s} - x'\tilde{T}_{\underline{D}}^{1-s}x\tilde{T}_{\underline{D}}^{s-1} = (z - \tilde{T}_{\underline{D}}^{-s}z\tilde{T}_{\underline{D}}^s) + (z'x' - x'\tilde{T}_{\underline{D}}z'\tilde{T}_{\underline{D}}^{-1})$$

where $z = x\tilde{T}_{\underline{D}}^s x'\tilde{T}_{\underline{D}}^{-s} \in H_n$ and $z' = \tilde{T}_{\underline{D}}^{-s}x\tilde{T}_{\underline{D}}^s \in H_n$. Hence it is enough to show that $\mathfrak{f}\eta\tilde{\omega}(z'x' - x'\tilde{T}_{\underline{D}}z'\tilde{T}_{\underline{D}}^{-1}) = 0$ (see 42.8(a)) and

(a) $\mathfrak{f}\eta\tilde{\omega}(z - \tilde{T}_{\underline{D}}^{-s}z\tilde{T}_{\underline{D}}^s) = 0$

for any $z \in H_n$. This follows from 41.6(c).

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Department of Mathematics, M.I.T., Cambridge, MA 02139